ELEMENTARY SURGERY MANIFOLDS AND THE ELEMENTARY IDEALS

J. P. NEUZIL

Abstract. We prove the following: If $M^3$ is a closed 3-manifold obtained by elementary surgery on a knot $K$ in $S^3$ and $H_1(M^3)$ is a nontrivial cyclic group, then the first elementary ideal $\pi_1(M^3)$ in the integral group ring of $H_1(M^3)$ is the principal ideal generated by the polynomial of $K$.

In this paper we study the 3-manifolds which are obtained by elementary surgery along a knot in $S^3$ and which are not homology spheres. This allows us to use the free calculus. Our main result is the following: If $M^3$ is a closed 3-manifold obtained by elementary surgery along a knot $K$ in $S^3$ and $H_1(M^3)$ is a nontrivial cyclic group $C$, then the first elementary ideal of $\pi_1(M^3)$ in the integral group ring of $C$ is the principal ideal generated by the first knot polynomial of $K$.

We will use the notation of the free calculus as developed in Chapter VII of [2]. For $n > 0$, we will use $\mathbb{Z}_n$ to denote the cyclic group of order $n$, and $\mathbb{Z}_0$ will be the infinite cyclic group. The generator of $\mathbb{Z}_n$ will be $t$. $J(\mathbb{Z}_n)$ will denote the integral group ring of $\mathbb{Z}_n$, thus the elements of $J(\mathbb{Z}_n)$ for $n > 0$, are finite sums $a_0 + a_1t + \cdots + a_{n-1}t^{n-1}$ where each $a_j$ is an integer. We will use $\tau$ to denote the trivializer of $J(\mathbb{Z}_n)$, that is, $\tau: J(\mathbb{Z}_n) \to \mathbb{Z}$ is defined by $\tau(t) = 1$, so that $\tau(f(t)) = f(1)$. For $n > 0$, $\Lambda_n$ will denote the element $\sum_{i=0}^{n-1} t^i$ of $J(\mathbb{Z}_n)$ and we let $\Lambda_0 = 0$. Note that $t^k \Lambda_n = \Lambda_n$ for all $k$, hence, for any $f(t) \in J(\mathbb{Z}_n), \ f(t) \Lambda_n = \tau(f(t)) \Lambda_n = f(1) \Lambda_n$. If $f(t) \in J(\mathbb{Z}_n)$, $(f(t))$ will denote the principal ideal generated by $f(t)$.

We use $\langle x_1, \ldots, x_k \mid R_1, \ldots, R_m \rangle$ to denote a group given by generators and relations. Let $G = \langle x_1, \ldots, x_k \mid R_1, \ldots, R_m \rangle$. Suppose the abelianization of $G$ is $\mathbb{Z}_n$. We use $\phi$ to denote the corresponding homomorphism of the free group $\langle x_1, \ldots, x_k \rangle$ onto $\mathbb{Z}_n$. Thus the Alexander matrix of $G$ has $\phi(\partial R_i / \partial x_j)$ as the entry in the $i$th row and the $j$th column. We let $E_n(G)$ denote the $n$th elementary ideal of $G$ as defined in [2, p. 101]. If $K$ is a knot in $S^3$, we let $E_n(\pi_1(S^3 - K)) = E_n(K)$. For a 3-manifold $M^3$, we let $E_n(\pi_1(M^3)) = E_n(M^3)$. Let $K$ be a knot in $S^3$ and let $N$ be a solid tubular neighborhood of $K$. Then $N$ is a solid torus. Let $(m, l)$ be a meridian-longitude pair for $N$. (This

Received by the editors March 15, 1976 and, in revised form, February 28, 1977.


© American Mathematical Society 1978
means that \( m \) and \( l \) are simple closed curves on \( \partial N \), \( m \) bounds a disk in \( N \) but not on \( \partial N \), and \( l \) is a homology generator of \( N \). In addition, assume that \( l \) is homologically trivial in \( \text{Cl}(S^3 - N) \). The elementary surgery manifold \( M^3(K; n, s) \) is constructed as follows: remove \( \text{Int} N \) from \( S^3 \) and sew in a new solid torus \( T \) so that a meridian of \( T \) is sewn to a curve \( C \) which is homologous to \( nm + sl \) on \( \partial N \). If \( n \neq 0 \), \( n \) and \( s \) must be relatively prime and if \( n = 0 \), we must have \( s = \pm 1 \). Note that \( H_1(M^3(K; n, s)) = \mathbb{Z}_m \); hence in this paper we restrict our attention to the case \( |n| \neq 1 \). Note also that \( \pi_1(M^3(K; n, s)) \) may be obtained by adding to \( \pi_1(S^3 - K) \) a relation which trivializes the element of \( \pi_1(S^3 - K) \) corresponding to the curve \( C \). Since \( M^3(K; n, s) \) and \( M^3(K; -n, -s) \) are homeomorphic, we will assume throughout that \( n \) is nonnegative. If \( J \) is a simple closed curve in a space \( X \), we will use \( J \) to denote the simple closed curve itself, the corresponding element of \( \pi_1(X) \) and the corresponding element of \( H_1(X) \). Finally, if \( S \) is a set of integers, we will use \( \text{GCD} S \) to denote the greatest common divisor of the elements in \( S \). The proofs of Theorems 1 and 2 given here were suggested to the author by the referee and represent a substantial improvement over the original proofs.

**Theorem 1.** The Alexander matrix of \( \pi_1(M^3(K; n, s)) \) is obtained from the matrix of \( \pi_1(S^3 - K) \) by adjoining a new row with one entry \( \Lambda_n \) and the rest zeros. In addition, the other entries in the column containing \( \Lambda_n \) are all zeros.

We should note here that when we say that the Alexander matrix is a certain matrix, we always mean up to equivalence as defined in [2, p. 101].

**Proof of Theorem 1.** Let \( G = \pi_1(S^3 - K) \) and let \( G' \) denote the commutator subgroup of \( G \). \( G \) has a presentation \( \langle a, x_1, \ldots, x_m | R_1, \ldots, R_m \rangle \), where \( a \) is a meridian of \( K \), \( x_j \in G' \) for \( 1 < j < m \), and \( \phi(\partial R_i/\partial a) = 0 \) for \( 1 < i < m \) [3, p. 415]. Hence \( G \) has Alexander matrix \( (\phi(\partial R_i/\partial x_j))_0 \). Now the matrix of \( \pi_1(M^3(K; n, s)) \) is obtained by adding a row \( (\phi(\partial S/\partial x_j), \phi(\partial S/\partial a)) \) where \( S \) is the relator \( a^n \). Since \( l \) is the boundary of a Seifert surface of \( K \), \( l \in G'' \), hence \( \phi(\partial l/\partial a) = \phi(\partial l/\partial x_j) = 0 \). Therefore, \( \phi(\partial S/\partial a) = \Lambda_n \) and \( \phi(\partial S/\partial x_j) = 0 \), hence the Alexander matrix of \( \pi_1(M^3(K; n, s)) \) is

\[
\begin{pmatrix}
\phi(\partial R_i/\partial x_j) & 0 \\
0 & \Lambda_n
\end{pmatrix}.
\]

This completes the proof of Theorem 1.

**Theorem 2.** The first elementary ideal \( E_1(M^3) \) of \( \pi_1(M^3(K; n, s)) \) is the principal ideal in \( J(\mathbb{Z}_n) \) generated by \( \Delta_1(t) \), the first knot polynomial of \( K \).

**Proof of Theorem 2.** The matrix obtained in Theorem 1 is an \( (m + 1) \times (m + 1) \) matrix and \( E_1(M^3) \) is generated by \( m \times m \) subdeterminants of this matrix. Hence, \( E_1(M^3) \) is generated by \( \Delta_1(t) \) and \( f_j(t)\Lambda_n \), \( 1 < j < \mu \), where \( f_1(t), \ldots, f_\mu(t) \) are the nontrivial \( (m - 1) \times (m - 1) \) subdeterminants of the Alexander matrix of \( K \), that is, the generators of \( E_2(K) \). Therefore,
\[ E_1(M^3) = (\Delta_1(t)) + \Lambda_n \cdot E_2(K). \]

But
\[ \Lambda_n \cdot E_2(K) = \Lambda_n \cdot \tau(E_2(K)) = \Lambda_n \cdot Z = (\Lambda_n). \]

Now, for any knot \( K \), \( \Delta_1(1) = \pm 1 \), hence
\[ \Lambda_n = (\Delta_1(1))^2 \Lambda_n = \Delta_1(t) \cdot \Delta_1(t) \cdot \Lambda_n \in (\Delta_1(t)). \]

Therefore,
\[ E_1(M^3) = (\Delta_1(t)) + (\Lambda_n) = \Delta_1(t). \]

This completes the proof of Theorem 2.

We finish with two corollaries to Theorem 2. The first is an alternate proof of Theorem 1 of [4].

**Corollary 1.** If \( K \) is a knot in \( S^3 \) with nontrivial polynomial \( \Delta_1(t) \) then \( M^3(K; n, s) \) is never topologically equivalent to \( S^2 \times S^1 \).

**Proof.** If \( M^3(K; n, s) = S^2 \times S^1 \) then \( H_1(M^3) = \pi_1(M^3) = \mathbb{Z}_0 \); hence \( n = 0 \) and \( s = \pm 1 \). But the first elementary ideal of \( J(\mathbb{Z}_0) \) of the infinite cyclic group \( \mathbb{Z}_0 \) is all of \( J(\mathbb{Z}_0) \) but, by Theorem 2, \( E_1(M^3) = (\Delta_1(t)) \) and \( (\Delta_1(t)) \neq J(\mathbb{Z}_0) \) since \( \Delta_1(t) \) is nontrivial. Hence \( \pi_1(M^3) \neq \pi_1(S^2 \times S^1) \), so \( M^3(K; n, s) \neq S^2 \times S^1 \).

Before stating the next corollary, we note that alternating knots with nontrivial polynomials satisfy the hypotheses. See [1] or [5].

**Corollary 2.** Suppose \( K \) is a knot in \( S^3 \) with polynomial \( \Delta_1(t) = a_0 + a_1t + a_2t^2 + \cdots + a_pt^p \). Let \( \alpha = a_0 + a_2 + a_4 + \cdots \), that is, \( \alpha \) is the sum of the coefficients of even powers of \( t \) in \( \Delta_1(t) \). If \( |\alpha| > 1 \) then \( \pi_1(M^3(K; n, s)) \) is never a finite cyclic group of even order.

**Proof.** In this proof, \( (\Delta_1(t))^n \) will denote the principal ideal in \( J(\mathbb{Z}_n) \) generated by \( \Delta_1(t) \). Now if \( \pi_1(M^3) \) is cyclic then \( \pi_1(M^3) = H_1(M^3) = \mathbb{Z}_0 \); hence it suffices to show that \( \pi_1(M^3(K; n, s)) \neq \mathbb{Z}_n \) for even \( n \). To show this it suffices to show that \( (\Delta_1(t))^n \neq J(\mathbb{Z}_n) \) for even \( n \), since the first elementary ideal in \( J(\mathbb{Z}_n) \) of \( \pi_1(M^3) \) is \( (\Delta_1(t))^n \) and the first elementary ideal in \( J(\mathbb{Z}_n) \) of \( \mathbb{Z}_n \) is all of \( J(\mathbb{Z}_n) \). But to show \( (\Delta_1(t))^n \neq J(\mathbb{Z}_n) \) for even \( n \), it suffices to show \( (\Delta_1(t))^2 \neq J(\mathbb{Z}_2) \), because, for even \( n \), there is a ring homomorphism of \( J(\mathbb{Z}_n) \) onto \( J(\mathbb{Z}_2) \) which takes \( (\Delta_1(t))^n \) onto \( (\Delta_1(t))^2 \).

Now suppose the contrary, that is, suppose \( (\Delta_1(t))^2 = J(\mathbb{Z}_2) \). In \( J(\mathbb{Z}_2) \), \( \Delta_1(t) = \alpha + (\varepsilon - \alpha)t \) where \( \varepsilon = \pm 1 \). Now if \( (\Delta_1(t))^2 = J(\mathbb{Z}_2) \), then there is an element \( f(t) \) of \( J(\mathbb{Z}_2) \) such that \( f(t)\Delta_1(t) = 1 \). Say \( f(t) = x + yt \) where \( x \) and \( y \) are integers. Then
\[ 1 = f(t)\Delta_1(t) = [\alpha x + (\varepsilon - \alpha)y] + [(\varepsilon - \alpha)x + ay]t; \]

hence \( \alpha x + (\varepsilon - \alpha)y = 1 \) and \( (\varepsilon - \alpha)x + ay = 0 \). Solving simultaneously, we obtain \( x = \alpha/(2\varepsilon - \alpha) \) which cannot be an integer unless \( |\alpha| < 1 \), which contradicts the hypothesis.
REFERENCES


DEPARTMENT OF MATHEMATICS, KENT STATE UNIVERSITY, KENT, OHIO 44242