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# THE PERFECT $n$TH POWER WHICH DIVIDES A NONZERO POLYNOMIAL

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ABSTRACT. In terms of rational operations, an algorithm is given to obtain the monic polynomial of maximal degree whose $n$th power divides a given nonzero polynomial.

Let $n$ be an integer $\geqslant 2$; let $F(X)$ be a nonzero polynomial in a single variable $X$ over a field $E$ of characteristic $p$ such that either $p = 0$ or $p > \deg(F(X))$; and let $\bar{E}$ be an algebraic closure of $E$. Then, there exist unique polynomials $G(X)$ and $H(X)$ over $E$ such that: $G(X)$ is monic,

$$F(X) = (G(X))^n H(X),$$

and the multiplicity of each root of $H(X)$ in $\bar{E}$ is less than $n$. We shall describe an algorithm to express $G(X)$ and $H(X)$ in terms of $F(X)$ by means of rational operations. There are applications in [1] for the special case where $n = 2$.

As a first step, we set $d_0(X) = F(X)$ and

$$d_k(X) = \text{g.c.d.}\left(d_{k-1}(X), F^{(k)}(X)\right) \quad \text{for } k = 1, 2, \ldots.$$

Since $d_k(X)$ is the greatest common divisor of $d_{k-1}(X)$ and the $k$th derivative $F^{(k)}(X)$ of $F(X)$, $d_k(X)$ is monic and there exists a least positive integer $m$ such that

$$d_k(X) = 1 \quad \text{for } k = mn, mn + 1, mn + 2, \ldots.$$

We set

$$q_k(X) = \frac{d_{nk}(X)(d_{nk+n}(X))^{n-1}}{(d_{nk+n-1}(X))^n} \quad \text{for } k = 0, 1, 2, \ldots. \tag{1}$$

LEMMA. *For $k = 0, 1, 2, \ldots, q_k(X)$ is a nonzero polynomial in $X$ over $E$ each of whose roots has multiplicity less than n. Moreover, for $r \neq s$, $q_r(X)$ and $q_s(X)$ are relatively prime.*

PROOF. There exists a factorization

$$d_{nk}(X) = \varepsilon_0(X - \theta_1)^{i_1} \cdots (X - \theta_\mu)^{i_\mu}\psi_k(X)$$

over $\bar{E}$ such that: $i_1, i_2, \ldots, i_\mu \geqslant n$; $\varepsilon_0 \neq 0$; $\psi_k(X)$ is a monic polynomial

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whose roots are distinct from $\theta_1, \theta_2, \ldots, \theta_\mu$, and each root of $\psi_k(X)$ has multiplicity less than $n$. (In case $\mu = 0$, the product

$$(X - \theta_1)^{i_1} \cdots (X - \theta_\mu)^{i_\mu}$$

is to be replaced by 1.) We obtain

$$d_{nk+n-1}(X) = (X - \theta_1)^{i_1 - (n-1)} \cdots (X - \theta_\mu)^{i_\mu - (n-1)},$$

$$d_{nk+n}(X) = (X - \theta_1)^{i_1 - n} \cdots (X - \theta_\mu)^{i_\mu - n},$$

and $q_k(X) = \varepsilon_0 \psi_k(X)$. Since the divisibility in (1) occurs over $E$, $q_k(X)$ is a polynomial in $X$ over $E$ each of whose roots in $\bar{E}$ has multiplicity less than $n$. If $\xi$ is a root of $q_k(X)$ in $\bar{E}$, then $\psi_k(\xi) = 0$, $d_{n(k+j)}(\xi) \neq 0$ for $j = 1, 2, \ldots$, and $q_{k+j}(\xi) \neq 0$ for $j = 1, 2, \ldots$. Thus, for $r \neq s$, $q_r(X)$ and $q_s(X)$ have no common roots. This completes the proof.

THEOREM. *The polynomials $G(X)$ and $H(X)$ defined over $E$ by*

$$G(X) = \prod_{k=1}^{m} \frac{d_{nk-1}(X)}{d_{nk}(X)} \quad \textit{and} \quad H(X) = \prod_{k=0}^{m} q_k(X)$$

*are such that $F(X) = (G(X))^n H(X)$, $G(X)$ is monic, and the multiplicity of each root of $H(X)$ in $\bar{E}$ is less than $n$.*

PROOF. For $k = 1, 2, \ldots$, both $d_{nk-1}(X)$ and $d_{nk}(X)$ are monic and $d_{nk}(X)$ divides $d_{nk-1}(X)$ over $E$. Thus, $G(X)$ is monic. By the lemma, the multiplicity of any root of $H(X)$ is less than $n$. Finally, we have $d_j(X) = 1$, for $j \geqslant mn$, and

$$(G(X))^n H(X) = \left[ \prod_{k=0}^{m} \frac{(d_{nk+n-1})^n}{(d_{nk+n})^n} \right] \left[ \prod_{k=0}^{m} \frac{d_{nk}(d_{nk+n})^{n-1}}{(d_{nk+n-1})^n} \right]$$

$$= \prod_{k=0}^{m} \frac{d_{nk}}{d_{nk+n}} = d_0 = F(X).$$

This completes the proof.

## REFERENCES

1. R. Chalkley, *Algebraic differential equations of the first order and the second degree*, J. Differential Equations **19** (1975), 70–79.

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