A SIMPLE NOETHERIAN RING NOT MORITA EQUIVALENT TO A DOMAIN

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Abstract. An example of Zalesskii and Neroslavskii is used to produce an example of a simple ring that is not Morita equivalent to a domain.

In [5] an example is given of a simple Noetherian ring with divisors of zero but without nontrivial idempotents. In this note we show that this example also gives a negative answer to the important question of whether simple Noetherian rings are Morita equivalent to domains, thus answering [1, Question 1, p. 113].

The ring is constructed in the following way. Let \( k \) be a field of characteristic two. Define \( R_1 = k(y)[X, X^{-1}] \) for indeterminates \( y \) and \( X \). Let \( g \) be the \( k(y) \)-automorphism of \( R_1 \), defined by \( g(X) = yX \) and let \( R_2 \) be the twisted group ring \( R_1 \langle g \rangle \); i.e. as an additive group, \( R_2 \) is isomorphic to the ordinary group ring but multiplication is defined by \( rg = grg \) for \( r \in R_1 \).

Let \( h \) be the \( k(y) \)-automorphism of \( R_2 \) defined by \( h(X) = X^{-1} \) and \( h(g) = g^{-1} \) and define \( S \) to be the twisted group ring \( R_2 \langle h \rangle \). The ring \( S \) was first constructed in [5], where the following was proved.

Theorem 1. \( S \) is a simple Noetherian ring, not a domain, such that the only idempotents of \( S \) are 0 and 1.

In [5], \( S \) was actually defined as a localisation of a group ring over \( k \). However, the characterisation given here is more convenient as it provides an easy method of calculating the Krull dimension of \( S \), written \( \text{Kdim } S \) (for our purposes the following definition suffices. Given a prime ring \( R \) then \( \text{Kdim } R = 1 \) if \( R \) is not Artinian but \( R/I \) is an Artinian module for any essential one-sided ideal \( I \)).

Theorem 2. \( \text{Kdim } S = 1 \).

Proof. Clearly \( R_1 \) is hereditary. Since \( g \) leaves no ideal of \( R_1 \) invariant, \( R_2 \) is hereditary by [3, Theorem 2.3]. Thus \( \text{Kdim } R_2 = 1 \) (see for example [2, Theorem 1.3]). But \( S \) is finitely generated as a left or right \( R_2 \)-module. So \( \text{Kdim } S < 1 \) and clearly we have equality.
This result enables us to use the results of [4] to show that \( S \) has the properties described in the title.

**Theorem 3.** \( S \) is a simple Noetherian ring that is not Morita equivalent to a domain.

**Proof.** Suppose \( S \) is Morita equivalent to a domain \( A \). Then \( \text{Kdim} \ A = 1 \) by Theorem 2. Let \( P \) be the image of \( S \) under the equivalence of the right module categories. Since \( S \) is not a domain, \( P \) is not isomorphic to a right ideal of \( A \). Thus, with \( \text{rk} \ P \) being the rank of the biggest free module that can be embedded in \( P \), we have \( \text{rk} \ P > 2 = 1 + \text{Kdim} \ A \). So by [4, Theorem 2.1], \( P \cong Q \oplus A \) for some nonzero module \( Q \). But then \( S \cong I \oplus J \) for some nonzero right ideals \( I \) and \( J \) of \( S \). This implies that \( S \) has nontrivial idempotents, which contradicts Theorem 1.

Since \( (1 + h)S \) has a periodic projective resolution, \( S \) has infinite global dimension. Thus it is still possible that any simple Noetherian ring of finite global dimension is Morita equivalent to a domain. Indeed, using [4], it is possible to show that a simple Noetherian ring \( R \), with finite global dimension and \( \text{Kdim} \ R = 1 \), is Morita equivalent to a domain.

**References**

5. A. E. Zalesskii and O. M. Neroslavskii, There exists a Noetherian ring with divisors of zero but without idempotents, Comm. Algebra 5 (3) (1977), 231–244. (Russian)