EXTREMAL HOLOMORPHIC IMBEDDINGS BETWEEN THE BALL AND POLYDISC

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Abstract. The following problem of Fornaess and Stout is considered: Find, among all polydiscs imbedded in the unit ball, the one which contains the largest ball centered at the origin.

Fornaess and Stout [1] have observed (as a consequence of their result that a monotone union of (imbedded) polydiscs in a taut complex manifold is (biholomorphic to) a polydisc) that an imbedded polydisc in the unit ball in $\mathbb{C}^m$ ($m > 1$) cannot contain balls centered at the origin of radius arbitrarily close to one. More precisely, there is a positive number $R_0(m) < 1$ such that the image of the unit polydisc $U^m$ by an imbedding in the unit ball $B_m$ in $\mathbb{C}^m$ never contains $sB_m$ for $s > R_0(m)$, where, for $W \subseteq \mathbb{C}^m$, $sW = \{sz : z \in W\}$. Nevertheless, $B_m$ contains an imbedded polydisc of full measure [1]. Fornaess and Stout, by exhibiting the map $F_0: U^m \rightarrow B_m$, $F_0(z) = \frac{z}{\sqrt{m}}$ with $F_0(U^m) \supseteq B_m/\sqrt{m}$, showed that $R_0(m) > 1/\sqrt{m}$ and raised the problem of finding the smallest (or any specific) value of $R_0(m)$. We show that, in fact, the smallest value of $R_0(m)$ is $1/\sqrt{m}$ and that $F_0$ is the unique extremal imbedding, up to automorphisms.

Proposition 1. If $F: U^m \rightarrow B_m$ is a holomorphic imbedding for which $F(U^m) \supseteq sB_m$, then $s < 1/\sqrt{m}$. Moreover, the equality $s = 1/\sqrt{m}$ holds if and only if $F = B \circ F_0 \circ A$ where $A$ is a biholomorphism of $U^m$ and $B$ is a unitary transformation.

Corollary 1. The polydisc $U^m/\sqrt{m} \subseteq B_m$ is maximal among imbedded polydiscs in $B_m$.

The reasoning of Fornaess and Stout also shows that there is a positive number $S_0(m) < 1$ such that if $B_m$ is imbedded into $U^m$, then the image does not contain $sU^m$ for $s > S_0(m)$. Now consider the inclusion map $G_0: B_m \rightarrow U^m$, $G_0(z) = z$. Then $G_0(B_m) \supseteq U^m/\sqrt{m}$ and so $S_0(m) > 1/\sqrt{m}$. This is the extremal case:

Proposition 2. If $G: B_m \rightarrow U^m$ is a holomorphic imbedding such that $G(B_m) \supseteq sU^m$, then $s < 1/\sqrt{m}$. Moreover the equality $s = 1/\sqrt{m}$ holds if and only if $G = G_0 \circ B(= B)$ where $B$ is a biholomorphism of $B_m$.
Corollary 2. The ball \( B_m \subseteq U^m \) is maximal among imbedded balls in the polydisc \( U^m \).

Proof of Proposition 1. Let \( \|z\| \) be the Euclidean norm for \( z \in \mathbb{C}^m \). Assume, first, that \( F(0) = 0 \). Write \( F \) in a vector Taylor series: \( F(z) = \sum a_a z^a \), where \( a_a \in \mathbb{C}^m \). The almost everywhere defined boundary values of the bounded holomorphic function \( F \) will also be denoted by \( F \). Since \( \lim \inf \| F(re^{i\theta}, 0, \ldots, 0) \| > s \) as \( r \uparrow 1 \), we have

\[
s^2 < \frac{1}{2\pi} \int_0^{2\pi} \| F(e^{i\theta}, 0, 0) \|^2 \; d\theta = \sum \| a_\alpha \|^2 : \alpha \in \mathbb{S}_1 \]

where \( \mathbb{S}_k = \{ \alpha = (\alpha_1, \ldots, \alpha_m) : \alpha_k > 0, \alpha_j = 0 \text{ for } j \neq k \} \) for \( k = 1, 2, \ldots, m \). In the same way we get \( s^2 < \sum \| a_\alpha \|^2 : \alpha \in \mathbb{S}_k \). Adding these \( m \) inequalities, we have

\[
ms^2 < \sum \| a_\alpha \|^2 : \alpha \in \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \cup \mathbb{S}_m \]

\[
< \sum \| a_\alpha \|^2 : \int_{T^m} \| F \|^2 \; dh < 1,
\]

where \( h \) is Haar measure on the torus \( T^m \); we are using \( a_\alpha = 0 \) for \( \alpha = 0 \) and \( \| F \|^2 < 1 \) a.e. on \( T^m \). Thus \( s < 1/\sqrt{m} \).

In the case of equality \( s = 1/\sqrt{m} \) we have (i) \( a_\alpha = 0 \) for \( \alpha \in \mathbb{S}_1 \cup \mathbb{S}_2 \cup \cdots \cup \mathbb{S}_m \), (ii) \( \| F(0, \ldots, e^{i\theta}, \ldots, 0) \| = 1/\sqrt{m} \) a.e. on the unit circle, where \( e^{i\theta} \) is in the \( k \)th position, for \( 1 \leq k \leq m \), and (iii) \( \| F \| = 1 \) a.e. on \( T^m \). By (i) and (ii) we can write \( F(z_1, z_2, \ldots, z_m) = F_1(z_1) + F_2(z_2) + \cdots + F_m(z_m) \) for \( F_k : U \to \mathbb{C}^m \) with \( F_k(0) = 0 \) and \( \| F_k(e^{i\theta}) \| = 1/\sqrt{m} \) a.e. on the circle; abusing notation, we shall also view \( F_k \) as a mapping defined on \( U^m \) which depends only on the \( k \)th variable.

For \( z, w \in \mathbb{C}^m \) put \( \langle z, w \rangle = \sum z_j w_j \) the standard Hermitian inner product. Then \( \Re \langle z, w \rangle \) is the standard real inner product on \( R^{2m} = \mathbb{C}^m \). Now (ii) and (iii) imply for almost all \( p = (e^{i\theta_1}, \ldots, e^{i\theta_m}) \in T^m \),

\[
1 = \| F(p) \|^2 = \sum \langle F_k(e^{i\theta_k}), F_j(e^{i\theta_j}) \rangle
\]

\[
= \sum m \| F_k(p) \|^2 + 2 \Re \sum_{k < j} \langle F_k(e^{i\theta_k}), F_j(e^{i\theta_j}) \rangle
\]

\[
= \sum m \frac{1}{m} + 2 Q(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_m})
\]

where \( Q(z_1, \ldots, z_m) = \Re \sum_{k < j} \langle F_k(z_k), F_j(z_j) \rangle \). Thus \( Q = 0 \) a.e. on \( T^m \). As \( Q \) is a bounded \( m \)-harmonic function on \( U^m \), it follows (see [2]) that \( Q \equiv 0 \) on \( U^m \). Thus \( \Re \langle F_k(z_k), F_j(z_j) \rangle = Q(0, \ldots, z_k, \ldots, z_j, \ldots, 0) = 0 \) for \( |z_j| < 1, |z_k| < 1 \); i.e., \( F_k(z_k) \) is real orthogonal to \( F_j(z_j) \). Define \( H_j \) to be the real linear span in \( \mathbb{C}^m \) of the set \( F_j(U) \). We have proved that the \( H_j \) are mutually (real) orthogonal. Since \( F \) is one-to-one, it follows that each \( F_j \) is one-to-one on \( U \). Therefore, \( \dim_{\mathbb{R}} H_j > 2 \). We conclude that \( \dim_{\mathbb{R}} H_j = 2 \) for each \( j \) and that \( H_j \) is complex linear (a "complex line"). Now, replacing \( F \) by \( V \circ F \)
where $V$ is a unitary transformation taking $H_j$ to the $j$th complex coordinate axis, we may assume that $H_j$ is the $j$th coordinate axis. Thus $F_j(z_j) = (0, \ldots, f_j(z_j), \ldots, 0)$ where, in the $j$th position, there is a bounded complex valued holomorphic function $f_j$ defined on $U$ and satisfying $f_j(0) = 0$ and $|f_j(e^{i\theta})| = 1/\sqrt{m}$ a.e.. Thus $\sqrt{m} f_j$ is an inner function. Since $f_j$ is one-to-one we conclude that $f_j(\xi) = e^{i\beta_j} \xi/\sqrt{m}$ for real $\beta_j$. This proves that $F = B \circ F_0$ where $B$ is a unitary transformation.

We have assumed that $F(0) = 0$. The general case can be reduced to this case by preceding $F$ by an automorphism of $U^m$.

**Proof of Corollary 1.** If $P$ is an imbedded polydisc in $B_m$ containing $U^m/\sqrt{m}$, then $P$ contains $B_m/\sqrt{m}$ and by Proposition 1, there is a unitary transformation $B$ such that $P = B(U^m/\sqrt{m})$. It follows that $B(T^m/\sqrt{m}) = T^m/\sqrt{m}$ and hence $|z_k| < 1/\sqrt{m}$ on $P$. Therefore $P = U^m/\sqrt{m}$.

**Proof of Proposition 2.** By preceding $G$ by an automorphism of $B_m$, we may assume, without loss of generality, that $G(0) = 0$. Let $\epsilon > 0$. Let $F = (G^{-1})_{|sU^m} : sU^m \to B_m$. Applying Proposition 1 to $F$, we see that $F(sU^m)$ does not contain $(1 + \epsilon)/\sqrt{m} B_m$. Thus there is a point $p \in B_m \setminus F(sU^m)$ with $||p|| < (1 + \epsilon)/\sqrt{m}$. Hence $q = G(p) \notin sU^m$ and so one of the coordinates, say $q_k$, of $q$ satisfies $|q_k| > s$. Schwarz’s lemma applied to the $k$th component $g_k$ of $G$ yields $|g_k(z)| < ||z||$ for all $z \in B_m$. Thus $s < |q_k| = |g_k(p)| < ||p|| < (1 + \epsilon)/\sqrt{m}$; i.e., $s < 1/\sqrt{m}$.

In order that $s = 1/\sqrt{m}$, $F(U^m/\sqrt{m})$ must contain $B_m/\sqrt{m}$. From Proposition 1 we conclude that $F$ is a unitary transformation (restricted to $U^m/\sqrt{m}$). This implies that $G$ has the desired form.

Corollary 2 follows directly from Proposition 2.

**References**


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