ON THE SEMI-CANONICAL PROPERTY IN THE PRODUCT SPACE $X \times I$

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Abstract. As one of the several properties in generalized metric spaces, the semi-canonical property has been discussed from the viewpoint of the extension of mappings. In this paper, that property will be discussed in product space $X \times I$ and reduced to a property of $X$.

1. Introduction. By a pair $(X, A)$ we mean a topological space $X$ with a closed subset $A$ of $X$. Let $(X, A)$ be a pair. As in [6], a collection $\mathcal{V} = \{V_\lambda: \lambda \in \Lambda\}$ of open subsets of $X$ is called a semi-canonical cover for $(X, A)$ if

1) $\bigcup_{\lambda \in \Lambda} V_\lambda = X - A$, and
2) for each $x \in A$ and each neighborhood $U$ of $x$ in $X$ there exists a neighborhood $W$ of $x$ in $X$ such that $\text{St}(W, \mathcal{V}) \subseteq U$, where

$$\text{St}(W, \mathcal{V}) = \bigcup \{ V \in \mathcal{V}: V \cap W \neq \emptyset \}$$

denotes the star of $W$ with respect to $\mathcal{V}$.

If a semi-canonical cover exists for $(X, A)$, $(X, A)$ is called a semi-canonical pair.

It was proved by D. Hyman ([6], [7]) that $(X, A)$ is a semi-canonical pair if $X$ is the image of a metric space by a closed continuous map. It is also mentioned by M. Cauty [3] that, if $X$ is a stratifiable space (cf. [2]), then any pair $(X, A)$ is semi-canonical. However, quite recently S. San-ou [11] pointed out that Cauty's statement was false by constructing an $M_1$-space $X$ (cf. [4]) such that $(X, A)$ was not semi-canonical for some closed subset $A$ of $X$.

The purpose of this paper is to discuss the semi-canonical property in the product space $X \times I$ of a $T_1$ space $X$ with the unit closed interval $I$ and to reduce it to a property in $X$.

Theorem 1. Let $X$ be a $T_1$ space. Then $(X \times I, X \times \{0\})$ is a semi-canonical pair if and only if $X$ is metrizable.

By Theorem 1 it can be easily seen that, if $X$ is any nonmetrizable $M_1$-space, then $X \times I$ is an $M_1$-space such that $(X \times I, X \times \{0\})$ is never semi-canonical.

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Theorem 2. Let $X$ be a $T_1$ space. Then $(X \times I, K \times \{0\})$ is a semi-canonical pair for each compact subset $K$ if and only if $X$ is a regular space which is a compact-covering, open image of a metric space.

Theorem 3. Let $X$ be a $T_1$ space. Then $(X \times I, \{(x, 0)\})$ is a semi-canonical pair for each point $x \in X$ if and only if $X$ is a regular, first countable space.

Throughout this paper, the following notations will be used: $X_0$ and $X_n$ denote the subspaces $X \times \{0\}$ and $X \times \{1/n\}$ of $X \times I$ for $n = 1, 2, \ldots$; $\pi$ denotes the projection from $X \times I$ onto $X$; and $I_n$ denotes the subspace $[0, 1/n]$ of $I$ for $n = 1, 2, \ldots$.

All spaces in this paper are $T_1$, and all maps are continuous.

2. Proof of Theorem 1. The sufficiency of the condition is clear, since every pair $(X, A)$ in a metric space $X$ is semi-canonical (cf. [6]). To prove necessity, suppose that there exists a semi-canonical cover $\mathcal{V}$ for $(X \times I, X_0)$. Put

$$\mathcal{V}_n = \pi(\mathcal{V}|X_n) = \{\pi(V \cap X_n) : V \in \mathcal{V}\}$$

for $n = 1, 2, \ldots$. Then $\{\mathcal{V}_n : n = 1, 2, \ldots\}$ is clearly a sequence of open covers of $X$.

Let us show that, for each point $x \in X$, the system $\{\text{St}(x, \mathcal{V}_n) : n = 1, 2, \ldots\}$ forms a neighborhood base at $x$, where $\text{St}(x, \mathcal{U})$ denotes the set $\text{St}(\text{St}(x, \mathcal{U}), \mathcal{U})$. Then $X$ is metrizable by a theorem of K. Morita [10]. To complete the proof, let $x$ be any point of $X$ and $G$ an arbitrary neighborhood of $x$ in $X$. Since $\mathcal{V}$ is a semi-canonical cover for $(X \times I, X_0)$, there exist a neighborhood $H_1$ of $x$ in $X$ and a positive integer $m$ such that $\text{St}(H_1 \times I_m, \mathcal{V}) \subset G \times I$ holds. Again, for the neighborhood $H_1$ of $x$ there exist a neighborhood $H_2$ of $x$ in $X$ and a positive integer $n$ such that $n > m$ and $\text{St}(H_2 \times I_n, \mathcal{V}) \subset H_1 \times I$. Now, let us show $\text{St}(x, \mathcal{V}_n) \subset G$. Pick an arbitrary point $y$ in $\text{St}(x, \mathcal{V}_n)$. Then there are two members $U, U'$ of $\mathcal{V}_n$ with $x \in U$, $y \in U'$ and $U \cap U' \neq \emptyset$. Let $z$ be a point of $U \cup U'$. By the definition of $\mathcal{V}_n$ there exist $V, V'$ in $\mathcal{V}$ such that

$$U = \pi(V \cap X_n) \quad \text{and} \quad U' = \pi(V' \cap X_n).$$

Hence, $(x, 1/n) \in V$, $(z, 1/n) \in V \cap V'$ and $(y, 1/n) \in V'$ hold. The first inclusion $(x, 1/n) \in V$ implies $V \subset H_1 \times I$, because $(x, 1/n)$ belongs to $H_2 \times I_n$; the second one implies $V' \subset G \times I$, because $(z, 1/n) \in V$ shows $(z, 1/n) \in H_1 \times I_n$ and $(z, 1/n) \in V'$ yields $V' \cap (H_1 \times I_n) \neq \emptyset$; and, as a consequence, the last inclusion $(y, 1/n) \in V'$ implies $y \in G$, which completes the proof.

3. Some lemmas.

Lemma 1. Let $X$ be a space. If the pair $(X \times I, \{(x, 0)\})$ is semi-canonical for every point $x \in X$, then $X$ is a regular space.

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1 A continuous map $f : X \to Y$ is called compact-covering if every compact subset of $Y$ is the image of some compact subset of $X$. 

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Proof. Using the same notations as in the proof of Theorem 1, it has been shown that, for a given point $x$ of $X$ and an arbitrary neighborhood $G$ of $x$, there exists an open cover $\mathcal{U}_n$ of $X$ such that $St^2(X, \mathcal{U}_n) \subset G$ holds. Clearly, $St(x, \mathcal{U}_n)$ is a neighborhood of $x$, whose closure is contained in $St^2(x, \mathcal{U}_n)$ and hence in $G$. This proves that $X$ is a regular space.

If $A \subset X$, then an $X$-base for $A$ is a collection $\mathcal{U}$ of open subsets of $X$ such that, if $x \in A$ and $V$ is a neighborhood of $x$ in $X$, then $x \in U \subset V$ for some $U \in \mathcal{U}$.

**Lemma 2.** Let $X$ be a regular $(T_1)$ space and $K$ a compact subset of $X$. If there exists a countable $X$-base for $K$, then there exists an $X$-base $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ for $K$ such that

1. $\mathcal{P}_n$ is a finite collection whose union covers $K$ for $n = 1, 2, \ldots$,
2. $\{P: P \in \mathcal{P}_{n+1}\}$ refines $\mathcal{P}_n$ for $n = 1, 2, \ldots$, and
3. for each point $x$ of $K$ and each neighborhood $G$ of $x$ in $X$, there exist a positive integer $n$ and a neighborhood $H$ of $x$ in $X$ such that $St(H, \mathcal{P}_n) \subset G$.

Proof. Let $\mathcal{B}$ be the given countable $X$-base for $K$. Since $\mathcal{B}|K$ is a countable base for $K$ itself, $K$ is metrizable. Hence, for any subset $E$ of $K$, the diameter $\delta(E)$ of $E$ is well defined and also, for any cover $\mathcal{S}$ of $K$, the mesh $\delta = \sup\{\delta(E); E \in \mathcal{S}\}$ is well defined.

For each $n$, let $\mathcal{U}_n$ be a finite subcollection of $\mathcal{B}$ such that

1. $\mathcal{U}_n$ covers $K$, and
2. mesh $\mathcal{U}_n[K] < 1/2^n$.

Let $\{\mathcal{V}_n: n = 1, 2, \ldots\}$ be the set of all finite subcollections of $\mathcal{B}$, each of which forms a minimal cover with respect to $K$; that is, any proper subcollection of $\mathcal{V}_n$ does not cover $K$ for $n = 1, 2, \ldots$. Put $\mathcal{W}_1 = \mathcal{U}_1 \land \mathcal{V}_1$ ($= \{U \cap V: U \in \mathcal{U}_1, V \in \mathcal{V}_1\}$) and $\mathcal{W}_{n+1} = \mathcal{W}_n \land \mathcal{U}_{n+1} \land \mathcal{V}_{n+1}$ for $n = 1, 2, \ldots$. Then each $\mathcal{W}_n$ is a finite collection of open subsets of $X$ whose union covers $K$.

Next, by induction on $n$, let us construct a finite collection $\mathcal{F}_n$ of closed subsets of $K$, a finite collection $\mathcal{P}_n$ of open subsets of $X$ and a function $\varphi_n$ from $\mathcal{F}_n$ onto $\mathcal{P}_n$ such that the following conditions are satisfied:

1. $\mathcal{F}_n$ is a closed cover of $K$ which refines $\mathcal{W}_n \land \mathcal{P}_{n-1}$, where $\mathcal{P}_0 = \{X\}$,
2. $\mathcal{P}_n$ refines $\mathcal{W}_n \land \mathcal{P}_{n-1}$,
3. if $F \in \mathcal{F}_n$, then $F \subset \mathcal{P}_n(F)$,
4. if $F \in \mathcal{F}_n$ and $F \subset 0 \in \bigcup_{i=1}^{n-1} \mathcal{P}_i \cup \bigcup_{i=1}^{n-1} (\mathcal{U}_i \cup \mathcal{V}_i)$, then $\varphi_n(F) = 0$,
5. and

Let $\mathcal{W}_1 = \{W_1, \ldots, W_k\}$. Since $\mathcal{W}_1$ covers $K$ and $K$ is normal, there exists a closed cover $\mathcal{F}_1 = \{F_1, \ldots, F_k\}$ of $K$ such that $F_i \subset W_i$ for $i = 1, \ldots, k$.

Hence $\mathcal{F}_1$ satisfies condition (3). Since $X$ is regular and $\mathcal{F}_1$ is a finite collection, each member of which is compact, and since $\mathcal{U}_1$ and $\mathcal{V}_1$ are also finite collections, it is easy to see that the function $\varphi_1$ and $\mathcal{P}_1 = \varphi_1(\mathcal{F}_1)$ are well defined to satisfy conditions (4)-(7), as well. The situation in each step
is the same as above, and thus \( \mathcal{P}_n \), \( \varphi_n \) and \( \mathcal{P}_n \) are all constructed quite similarly.

Now, it remains to show that the sequence \( \{ \mathcal{P}_n : n = 1, 2, \ldots \} \) is the required one in Lemma 2. Since \( \mathcal{P}_n \) is finite and satisfies (3) and (5), \( \mathcal{P}_n \) satisfies the condition (1). By (3) and (6), \( \mathcal{P}_n \) satisfies the condition (2). To prove that \( \{ \mathcal{P}_n : n = 1, 2, \ldots \} \) satisfies the condition (3), let \( x \) be any point of \( K \) and \( G \) an arbitrary neighborhood of \( x \) in \( X \). Since \( \mathcal{H} \) is an \( X \)-base for \( K \), there exists a \( B_0 \in \mathcal{H} \) such that \( x \in B_0 \subseteq G \).\(^2\) Let \( \mathcal{V} \) be a finite subcollection of \( \mathcal{H} \) which is a minimal cover with respect to \( K \) and which keeps \( B_0 \) as the only member of \( \mathcal{V} \) containing \( x \). Since \( K \) is a compact \( T_2 \) space and since \( \mathcal{H} \) is an \( X \)-base for \( K \), such \( \mathcal{V} \) certainly exists; further, for some \( n \), \( \mathcal{V} = \mathcal{V}_n \).

Let \( F_0 \in \mathcal{F}_n \) be a member with \( x \in F_0 \). Then \( F_0 \subseteq B_0 \) holds, because \( \mathcal{F}_n \) refines \( \mathcal{H}_0 \) which refines \( \mathcal{V}_n \) and \( B_0 \) is the only member of \( \mathcal{V} \) containing \( x \); and also, by (5) and (6), the inclusions \( F_0 \subseteq \varphi_n(F_0) \subseteq B_0 \) hold. Since \( \varphi_n(F_0) \) is an open set containing \( x \), there exists a positive integer \( m \) such that \( m > n \) and \( d(x, K - \varphi_n(F_0)) > 1/2^m \), where \( d \) denotes the metric function on \( K \). Since \( \mathcal{F}_{m+1} \) is a cover of \( K \) by (3)\( m+1 \), there exists an \( F_1 \in \mathcal{F}_{m+1} \) containing \( x \). To complete the proof, it suffices to show that

\[
\text{St}(\varphi_{m+1}(F_1), \mathcal{P}_{m+1}) \subseteq \varphi_n(F_0),
\]

because \( \varphi_{m+1}(F_1) \) is an open set in \( X \) containing \( x \) and \( \varphi_n(F_0) \) is contained in \( B_0 \), which is contained in \( G \). Let \( P \) be an arbitrary member of \( \mathcal{P}_{m+1} \) and \( F \) the corresponding member of \( \mathcal{F}_{m+1} \) by \( P = \varphi_{m+1}(F) \). If \( P \cap \varphi_{m+1}(F_1) \neq \emptyset \), then by (7)\( m+1 \), \( F \cap F_1 \neq \emptyset \). Since \( \mathcal{F}_{m+1} \) refines \( \mathcal{P}_m \) by (5)\( m+1 \) and \( \mathcal{P}_{m+1} \) refines \( \mathcal{H}_{m+1} \) by (4)\( m+1 \), and since \( \mathcal{H}_{m+1} \) refines \( \mathcal{P}_m \) whose mesh restricting to \( K \) is less than \( 1/2^{m+1} \), the diameter \( \delta(F \cup F_1) \) is less than \( 1/2^m \). Since \( x \) belongs to \( F_1 \), by the choice of \( m \), \( F \cup F_1 \subseteq \varphi_n(F_0) \) holds. Again by (6)\( m+1 \), \( \varphi_{m+1}(F) \subseteq \varphi_n(F_0) \) and thus \( P \subseteq \varphi_n(F_0) \) holds, which completes the proof.

**Lemma 3.** Let \( X \) be a regular \( (T_1) \) space and \( K \) a compact subset of \( X \). If there exists a countable \( X \)-base for \( K \), then \( (X, K) \) is a semi-canonical pair.

**Proof.** Let \( \bigcup_{n=1}^{\infty} \mathcal{P}_n \) be an \( X \)-base for \( K \) obtained by Lemma 2. For each \( n \), put \( G_n = \bigcup \{ P : P \in \mathcal{P}_n \} \). Then, by conditions (1) and (2) in Lemma 2, \( G_{n+1} \subseteq G_n \) for \( n = 1, 2, \ldots \) and \( K \subseteq \bigcap_{n=1}^{\infty} G_n \), and by condition (3) and by the fact that \( K \) is compact, it is easily seen that \( K = \bigcap_{n=1}^{\infty} G_n \).

Now, put \( \mathcal{V}_0 = \{ X - G_2 \} \) and \( \mathcal{V}_n = \mathcal{P}_n \{ G_n - G_{n+2} \} \) for \( n = 1, 2, \ldots \), and put \( \mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n \). Then it will be shown that \( \mathcal{V} \) is a semi-canonical cover for \( (X, K) \). Clearly, \( \mathcal{V} \) is an open cover of \( X \) - \( K \). To complete the proof, let \( x \) be any point of \( K \) and \( U \) an arbitrary neighborhood of \( x \) in \( X \). By condition (3) in Lemma 2, there exist a positive integer \( n \) and a neighborhood \( H \) of \( x \) in \( X \) such that \( \text{St}(H, \mathcal{P}_n) \subseteq U \). Put \( W = H \cap G_{n+1} \). Then \( W \) is a neighborhood of \( x \) in \( X \) such that \( W \cap V = \emptyset \) for each \( V \in \bigcup_{n=1}^{\infty} \mathcal{V}_n \).

\(^2\) If \( K \) is singleton, then \( \bigcup_{n=1}^{\infty} \mathcal{P}_n \) is easily chosen from the given countable \( X \)-base for \( K \), because \( X \) is regular. So, assuming that \( K \) is not a singleton, \( B_0 \) is picked out from \( \mathcal{H} \) such that \( K - B_0 \neq \emptyset \).
Therefore

\[ \text{St}(W, \mathcal{V}) = \text{St}(W, \bigcup_{i \geq n} \mathcal{V}_i) \subset \text{St}(W, \bigcup_{i \geq n} \mathcal{P}_i) \subset \text{St}(H, \mathcal{P}_n) \subset U \]

by condition (2) in Lemma 2, and that completes the proof.

4. Proofs of Theorems 2 and 3. The following characterization of the compact-covering open images of metric spaces, due to E. Michael and K. Nagami [9] will be used in the proof of Theorem 2.

**Theorem M-N (E. Michael and K. Nagami).** For a \( T_2 \) space \( X \), the following conditions are equivalent:

1. \( X \) is the compact-covering open image of a metric space.
2. Every compact subset of \( X \) is metrizable and of countable character in \( X \).
3. Every compact subset of \( X \) has a countable \( X \)-base.

**Proof of Theorem 2. Necessity.** Let \((X \times I, K \times \{0\})\) be a semi-canonical pair for any compact subset \( K \) of \( X \). Then \( X \) is a regular space by Lemma 1 putting \( K \) in the assumption a singleton. Next, it will be shown that each compact subset \( K \) of \( X \) has a countable \( X \)-base. Then \( X \) is the compact-covering open image of a metric space by Theorem M-N.

To complete the proof, let \( K \) be a compact subset of \( X \). By the assumption, there exists a semi-canonical cover \( \mathcal{V} \) for \((X \times I, K \times \{0\})\). Put \( \mathcal{V}_n \) the finite subcollection of \( \mathcal{V} \) which covers \( K \times \{1/n\} \), and put \( \mathcal{U}_n = \pi(\mathcal{V}_n | X_n) \) for \( n = 1, 2, \ldots \).

Then it is easy to show that the collection \( \bigcup_{n=1}^{\infty} \mathcal{U}_n \) is the required \( X \)-base for \( K \), by the same technique as in the proof of Theorem 1.

**Sufficiency.** It is easy to check that, if \( X \) is the compact-covering open image of a metric space, then so is \( X \times I \). Hence, for any compact subset \( K \) of \( X \), \( K \times \{0\} \) has a countable \( X \times I \)-base by Theorem M-N, and thus \((X \times I, K \times \{0\})\) is a semi-canonical pair by Lemma 3, which completes the proof.

**Proof of Theorem 3. Necessity.** By Lemma 1, \( X \) is a regular space. The first countability of \( X \) is proved by the same technique as in the proof of the necessity in Theorem 2, replacing \( K \) by a singleton.

**Sufficiency.** If \( X \) is a regular \( (T_1) \) first countable space, then so is \( X \times I \). In general, it is easily seen that, in any regular \( (T_1) \) first countable space \( Y \), the pair \((Y, \{y\})\) is always semi-canonical for each point \( y \in Y \). This completes the proof.

5. Comments. 1. From the proofs of Theorems 1, 2 and 3, it is easy to see that, in the conditions of these theorems, the closed interval \( I \) may be
replaced by any space containing a convergent sequence. By such replace-
ment in Theorem 1, one obtains a slight modification of the proof of the
following theorem due to D. M. Hyman [7], remembering two facts: (1) The
closed image of a metric space is a Fréchet-Urysohn space (cf. [8]); and (2)
any pair \((X, A)\) is semi-canonical if \(X\) is the closed image of a metric space
(cf. [7]).

**Theorem (D. Hyman).** If \(X \text{ and } Y\) are nondiscrete spaces and if \(X \times Y\) is
the closed image of a metric space, then \(X \text{ and } Y\) are metrizable.

2. The semi-canonical property need not be two-productive. For example,
let \(X = N \cup \{p\}\) be a subspace of Stone-Cech compactification \(\beta N\) of \(N\)
\((= \{1, 2, \ldots \})\) with \(p \in \beta N - N\). Then it is well known that \(X\) is not first
countable at \(p\), and thus \((X \times I, \{(p, 0)\})\) is not semi-canonical by Theorem
3. However, it is easy to see that any pair \((X, A)\) is always semi-canonical.

This example also shows that, in the conditions of Theorems 1 and 2,
\(X \times I\) cannot be replaced by \(X\). Clearly, then, the semi-canonical property in
\(X\) is very different from the semi-canonical property in \(X \times I\).

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