MAXIMAL SUBGROUPS OF PRIME INDEX IN A FINITE SOLVABLE GROUP

PAUL VENZKE

Abstract. In this note we show that a maximal subgroup of a finite solvable group has prime index if and only if it admits a cyclic supplement which permutes with one of its Sylow systems. In particular, a finite solvable group is supersolvable if and only if each maximal subgroup admits a cyclic supplement which permutes with a Sylow system of the maximal subgroup.

System quasinormalizers were introduced in [3] as a generalization of P. Hall's system normalizers. A system quasinormalizer of a finite solvable group $G$ is defined to be the subgroup generated by all cyclic subgroups of $G$ which permute with each element of a fixed Sylow system of $G$. In this note we use system quasinormalizers to characterize the maximal subgroups of prime index in a finite solvable group. As a consequence of this characterization, a sharpening of a result by Kegel is achieved. More specifically, Kegel [2] has shown that the class of finite groups which have the property that each maximal subgroup admits a cyclic supplement of prime power order is slightly larger than the class of all finite supersolvable groups. It will be shown here that the class of finite supersolvable groups is precisely the class of all finite solvable groups with the property that each maximal subgroup admits a cyclic supplement which permutes with each element in a Sylow system of the maximal subgroup. In this note we will be concerned only with finite solvable groups.

For a finite solvable group $G$ and a set of primes $\pi$, $G_{\pi}$ denotes a Hall $\pi$-subgroup of $G$ while $G^\pi$ denotes a Hall $\pi$-complement of $G$. In the case when $\pi$ consists of a single prime $p$, we write simply $G_p$ and $G^p$. By a Sylow system of $G$ is meant a complete set of permuting Hall subgroups of the group $G$. Let $\Sigma$ denote a Sylow system of the solvable group $G$. If $H$ is a subgroup of $G$ and $\{G_{\pi} \cap H : G_{\pi} \in \Sigma\}$ is a Sylow system of $H$, we say $\Sigma$ reduces into $H$ and denote this Sylow system of $H$ by $\Sigma \cap H$.

Let $\mathcal{S}$ be a Sylow system of a subgroup $H$ of the group $G$. The system quasinormalizer of $\mathcal{S}$ in $G$, denoted $N^G_G(\mathcal{S})$, is the subgroup of $G$ generated by all cyclic subgroups of $G$ which permute with each element of $\mathcal{S}$. For an element $x$, if $\langle x \rangle$ permutes with each element of $\mathcal{S}$, we will say that $x$ is $\mathcal{S}$-quasinormal in $G$.

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Lemma. Let $G$ be a finite solvable group and $M$ a maximal subgroup of $G$ with prime index $p$. If $\mathcal{S}$ is a Sylow system of $M$ and $\Sigma$ is a Sylow system of $G$ with $\Sigma \cap M = \mathcal{S}$, then $N^G_\mathcal{S}(\Sigma) \subseteq N^G_\Sigma(\mathcal{S})$.

Proof. In [3] it is observed that $N^G_\mathcal{S}(\Sigma)$ is generated by the elements of prime power order which are $\Sigma$-quasinormal. It suffices then to show that all such elements lie in $N^G_\Sigma(\mathcal{S})$. Let $y$ be a $\Sigma$-quasinormal element of $G$ with $|y| = r^n$ for some prime $r$.

Suppose $r \neq p$. Let $G^p$ be the Sylow $p$-complement of $G$ in $\Sigma$. As $\langle y \rangle$ permutes with $G^p$, $y \in G^p$. In that $\Sigma \cap M = \mathcal{S}$, $G^p = M^p \in \mathcal{S}$ and $y \in M$. For $M_\mathcal{S} \in \mathcal{S}$ we have $G_\mathcal{S} \cap M = M_\mathcal{S}$ where $G_\mathcal{S}$ lies in $\Sigma$. $\langle y \rangle G_\mathcal{S}$ is a group, and by the Dedekind identity, $\langle y \rangle G_\mathcal{S} \cap M = \langle y \rangle (G_\mathcal{S} \cap M) = \langle y \rangle M_\mathcal{S}$. Hence $\langle y \rangle$ permutes with each $M_\mathcal{S}$ in $\mathcal{S}$ so that $y \in N^G_\mathcal{S}(\mathcal{S})$. We may now assume $r = p$.

Let $M_q$ be a Sylow $q$-subgroup in $\mathcal{S}$ and $G_q$ a Sylow $q$-subgroup in $\Sigma$ so that $G_q \cap M = M_q$. For $q \neq p$, $M_q = G_q$, and since $y$ is $\Sigma$-quasinormal, $\langle y \rangle$ permutes with $M_q$. For $p = q$, $|G_p : M_p| = |G : M| = p$ so that $M_p$ is a normal subgroup of $G_p$. As $\langle y \rangle$ is a $p$-group and permutes with $G_p$, $y$ must lie in $G_p$. That is, $y \in N_G(M_p)$ so that $\langle y \rangle$ also permutes with $M_p$. Thus $\langle y \rangle$ permutes with each Sylow subgroup of $\mathcal{S}$ and so must permute with every element in $\mathcal{S}$. Therefore in this case also $y \in N^G_\mathcal{S}(\mathcal{S})$ and the lemma is established.

Theorem. Let $G$ be a finite solvable group and $M$ a maximal subgroup of $G$ with Sylow system $\mathcal{S}$. $M$ has a prime index in $G$ if and only if $G = MN^G_\mathcal{S}(\mathcal{S})$.

Proof. Suppose first that $M$ has prime index $p$. Let $\Sigma$ be a Sylow system of $G$ with $\Sigma \cap M = \mathcal{S}$. $M$ must complement a chief factor of $G$ which has prime order $p$ and $N^G_\mathcal{S}(\mathcal{S})$ covers this chief factor [3, Corollary 2.3]. Hence $G = MN^G_\mathcal{S}(\mathcal{S})$, so that by the preceding lemma the result follows.

On the other hand suppose $G = MN^G_\mathcal{S}(\mathcal{S})$ and let $G$ be the minimal counterexample to the theorem. We may thus assume that $M$ is core free. As $G = MN^G_\mathcal{S}(\mathcal{S})$, there are $\mathcal{S}$-quasinormal elements of $G$ which do not lie in $M$. Let $y$ be an element of smallest possible order having this property. By the maximality of $M$, $G = \langle y \rangle$. Thus $\langle \langle y \rangle \cap M \rangle G \subseteq M$, and since $M$ is core free, $\langle y \rangle \cap M = \{1\}$. Thus $|y| = |G : M| = p^a$ for some prime $p$.

Let $M_p$ be the Sylow $p$-subgroup of $M$ in $\mathcal{S}$. Since $\langle y \rangle$ permutes with $M_p$, $G_p = M_p\langle y \rangle$ is a Sylow subgroup of $G$. Let $T$ be a maximal subgroup of $G_p$ with $T \supseteq M_p$. Since $T = M_p(\langle y \rangle \cap T)$, it follows that $p = |G_p : T| = |\langle y \rangle : (\langle y \rangle \cap T)|$ and $\langle y \rangle \cap T = \langle y^p \rangle$. Hence $T = M_p\langle y^p \rangle$ and $\langle y^p \rangle$ permutes with $M_p$.

For $M_q$ a Sylow $q$-subgroup in $\mathcal{S}$ with $q \neq p$, $M_q\langle y \rangle$ is $p$-supersolvable. If $L$ is a maximal subgroup of $M_q\langle y \rangle$ containing $M_q$, then $L = M_q(\langle y \rangle \cap L)$. By the $p$-supersolvability of $M_q\langle y \rangle$, $p = |M_q\langle y \rangle : L| = |\langle y \rangle : \langle y \rangle \cap L|$ so that $\langle y \rangle \cap L = \langle y^p \rangle$. Thus $L = M_q\langle y^p \rangle$ and $\langle y^p \rangle$ permutes with $M_q$.

As we have shown that $\langle y^p \rangle$ permutes with each Sylow subgroup in $\mathcal{S}$, it follows that $y^p$ is $\mathcal{S}$-quasinormal. By the choice of $y$ we must have that
$y^p \in M$ and so lies in $\langle y \rangle \cap M = \{1\}$. It now follows that $|y| = |G : M| = p$ and the theorem follows.

In that a group is supersolvable if and only if each maximal subgroup has prime index [1], we have

**Corollary.** The finite solvable group $G$ is supersolvable if and only if for each maximal subgroup $M$ of $G$, $G = MN_G(\mathfrak{S})$, where $\mathfrak{S}$ is some Sylow system of $M$.

This may be restated as

**Corollary.** The finite solvable group $G$ is supersolvable if and only if every maximal subgroup $M$ of $G$ admits a cyclic supplement which permutes with each element in some Sylow system of $M$.

**References**


**Division of Science and Mathematics, Minot State College, Minot, North Dakota** 58701