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COMPACT σ-METRIC HAUSDORFF SPACES ARE SEQUENTIAL

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ABSTRACT. It is shown that a locally countably compact, regular T_2 space which may be covered by a countable number of metrizable subspaces is sequential.

In response to a question asked by A. V. Arhangel'skiĭ at the recent Prague Topology Conference (1976) we prove the assertion in the title. In fact we do slightly more (for definitions see below).

THEOREM 1. *Let X be a regular (Hausdorff) space which is locally countably compact. Suppose that $X = \bigcup_{n\in\omega} M_n$ where each M_n is metrizable in its subspace topology, then X is sequential.*

Professors M. E. Rudin and K. Kunen both pointed out that X is also sequentially compact (if X is countably compact). We shall give a proof here (Theorem 3) for completeness' sake and because we need their result.

DEFINITIONS 1. A set A in a space X is said to be *sequentially closed* if every convergent sequence of points from A has its limit in A. A space is *sequential* if all sequentially closed sets are in fact closed (thus e.g. $[0, \omega_1]$ is not sequential in the order topology because $[0, \omega_1)$ is sequentially closed but not closed). See S. P. Franklin [5], for the theory of sequential spaces and also Arhangel'skiĭ and Franklin [4].

A space is *countably compact* if every infinite set has a point of accumulation; it is *sequentially compact* if every sequence contains a convergent subsequence.

σ-metric spaces (i.e. those representable as a countable union of metrizable subspaces) arise naturally as images of complete metric spaces under closed mappings (Arhangel'skiĭ [1] and [2]. They were also considered in passing in [3, p. 118]).

DEFINITIONS 2. If p is a point of a space X we denote by $\bar{\mathcal{G}}_\delta(p)$ the family of sets which can be written in the form $\bigcap_{n\in\omega} G_n$ where: each G_n is open and contains p; cl G_1 is countably compact and cl $G_{n+1} \subseteq G_n$ for all n. It is important to observe that $\bar{\mathcal{G}}_\delta(p)$ is closed under countable intersection. In a countably compact, normal space $\bar{\mathcal{G}}_\delta(p)$ is the family of closed $\mathcal{G}_\delta$ sets containing p. We shall say that a set $N \subseteq X$ is *nonnegligible* for p (or just

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nonnegligible when p is fixed) if every H in $\bar{\mathcal{G}}_\delta(p)$ meets N. Otherwise it is *negligible*.

LEMMA 1. *Let $X = \bigcup_{n\in\omega} X_n$ be a regular space and suppose each point of X_n is a $\mathcal{G}_\delta$ point of X_n. Let A be sequentially closed in X and suppose that A is not closed. Then for each $p \in \operatorname{cl} A \setminus A$, the set A is nonnegligible for p.*

PROOF. Suppose otherwise. We assume for convenience of notation that $X = \operatorname{cl} A$. Thus for some $p_0 \in X \setminus A$ there is $H(p_0) \in \bar{\mathcal{G}}_\delta(p_0)$ with $H(p_0) \cap A = \varnothing$. For each $i \in \omega$ and each $p \in M_i$ choose open sets $U_n(p)$ such that $\operatorname{cl} U_{n+1}(p) \subseteq U_n(p)$ and $M_i \cap \bigcap_n U_n(p) = \{p\}$. Thus $H(p_0) \cap \bigcap_n U_n(p) \in \bar{\mathcal{G}}_\delta(p)$ provided $p \in H(p_0)$. We now derive a contradiction by defining by transfinite induction sets H_β integers i_β and points p_β for each $\beta < \omega_1$ so that

(i) $H_\beta \cap M_{i_\beta} = \{p_\beta\}$ and $H_\beta \in \bar{\mathcal{G}}_\delta(p_\beta)$,

(ii) $\beta < \beta' \to H_{\beta'} \subseteq H_\beta$,

(iii) $\beta < \beta' \to p_\beta \notin H_{\beta'}$.

It follows from this that $\{i_\alpha\colon \alpha < \omega_1\}$ is uncountable! Put $H_0 = H(p_0) \cap \bigcap_{n\in\omega} U_n(p_0)$. Then $H_0 \cap M_{i_0} = \{p_0\}$ where $p_0 \in M_{i_0}$. Now suppose that H_β, p_β, i_β have been defined for all $\beta < \alpha$ so that (i)–(iii) hold. If α is a limit ordinal then $H = \bigcap_{\beta<\alpha} H_\beta$ is nonempty and disjoint both from A and from each M_{i_β} for $\beta < \alpha$. Let $p_\alpha \in H$ and suppose that $p_\alpha \in M_{i_\alpha}$. Take $H_\alpha = H \cap \bigcap_n U_n(p_\alpha)$. Then $H_\alpha \in \bar{\mathcal{G}}_\delta(p_\alpha)$ and clearly (i)–(iii) hold.

If $\alpha = \gamma + 1$ observe that $H_\gamma = \{p_\gamma\}$ would imply that p_γ has countable character and hence that p_γ is the limit of a sequence in A contradicting $H_0 \cap A = \varnothing$. So for some $i_\alpha \neq \gamma$, $H_\gamma \cap M_{i_\alpha} \neq \varnothing$. Clearly $i_\alpha \notin \{i_\beta\colon \beta < \alpha\}$ by (i) and (iii). Let $p_\alpha \in M_{i_\alpha} \cap H_\gamma$. Choose an open set $U^\alpha \ni p_\alpha$ with $p_\gamma \notin U^\alpha$. We may assume that $U_1(p_\alpha)$ is chosen so that $U_1(p_\alpha) \cap M_{i_\alpha} \subseteq U^\alpha \cap M_{i_\alpha}$. Put $H_\alpha = H_\gamma \cap \bigcap_n U_n(p)$. Then H_α satisfies $H_\alpha \cap M_{i_\alpha} = \{p_\alpha\}$, $H_\alpha \in \bar{\mathcal{G}}_\delta(p_\alpha)$, $H_\alpha \cap \{p_\beta\colon \beta < \alpha\} = \varnothing$. This proves the lemma.

REMARK. The transfinite induction above can be avoided on the assumption that for each n every point of X_n has *countable character* in X_n (i.e. in the space X_n each point has a countable base for its neighbourhood system). For, if $p \in \operatorname{cl}(A) \setminus A$ and $H \in \bar{\mathcal{G}}_\delta(p)$ we may choose open sets G_n with $\operatorname{cl} G_1$ countably compact so that $H = \bigcap_n G_n$ and so that $\operatorname{cl} G_{n+1} \subseteq G_n$ for every n. Now pick $a_n \in G_n \cap A$. By Theorem 3 applied to $\operatorname{cl} G_1$ the sequence $\{a_n\}$ has a convergent subsequence converging to a point a, necessarily in A (since A is sequentially closed). But $a \in \operatorname{cl} U_n$, all n. Thus $a \in H \cap A \neq \varnothing$. Thus A is nonnegligible for p. I am grateful for this argument to the referee and also to R. Pol who also noticed this proof.

The argument of the lemma was motiviated by the following result.

THEOREM 2. *Let X be a regular, locally countably compact space. If X is the union of two metric spaces then X is a Fréchet-Urysohn space (i.e. for any $A \subseteq X$ and $p \in \operatorname{cl} A$ there is a sequence in A converging to p).*

PROOF. Let $X = M_0 \cup M_1$ and suppose that, contrary to theorem, there is $A \subseteq X$ and $p_0 \in \operatorname{cl} A$ so that no sequence in A converges to p_0. We may

assume that $X = \mathrm{cl}\, A$, that $p_0 \in M_0$ and that $A = M_1$ (otherwise consider $X = \mathrm{cl}(M_1 \cap A)$ or $X = \mathrm{cl}(M_0 \cap A)$). Now choose $H \in \bar{\mathcal{G}}_\delta(p_0)$ with $H \cap M_0 = \{p_0\}$. Since p_0 does not have countable character we see that p_0 cannot be isolated in H, and hence that $H \cap M_1$ is not compact. This implies that $H \cap M_1$ contains a discrete set D that is closed in $H \cap M_1$. By countable compactness of H, p_0 is the unique accumulation point of D and so D converges to p_0 (otherwise for some open U with $p_0 \in U$ we should find that $D \setminus U$ is infinite and cannot accumulate at p_0 nor on $H \cap M_1$).

A space X which is the union of three metric spaces but is not Fréchet-Urysohn may be constructed as follows. Let $\mathcal{F} = \{F_t\colon t \in (0, 1)\}$ be a maximal family of infinite, almost disjoint subsets of the natural numbers $\mathbf{N}$. The smallest topology on $\mathbf{N} \cup (0, 1)$ for which F_t converges to t is a locally-compact Hausdorff space whose one-point compactification is the desired X (by maximality of $\mathcal{F}$ each sequence in $\mathbf{N}$ contains a subsequence converging to some point of $(0, 1)$, although $\mathrm{cl}\, \mathbf{N} = X$).

Observe, however, that if we define for $A \subseteq X$, $A^{(1)} = \{x \in X\colon$ a sequence from A converges to $x\}$ and inductively $A^{(m)} = (A^{(m-1)})^{(1)}$ where $A^{(0)} = A$, then, as M. E. Rudin points out, Theorem 2 may readily be generalized (e.g., using the argument after the proof of Lemma 6 below) to an X which is the union of n metric spaces to read: for each $A \subseteq X$, $\mathrm{cl}\, A = A^{(n-1)}$.

Convention. Henceforth we assume that X satisfies the hypotheses of Theorem 1 and that, for some sequentially closed set A, a chosen point p in $(\mathrm{cl}\, A) \setminus A$ is held fixed. Thus for all $H \in \bar{\mathcal{G}}_\delta(p)$ we have $H \cap A \neq \varnothing$ and in fact $H \cap A$ is itself nonnegligible. We denote by seq-cl(B) the *sequential closure* of B (see [4], [5] and [6]). We fix a metric ρ_i on M_i.

Lemma 2. *If H is countably compact, then every sequentially closed subset of $A \cap H$ is sequentially compact.*

Proof. Immediate from Theorem 3 below.

Lemma 3. *If $L \subseteq X \setminus \{p\}$ is a Lindelöf subset, then L is negligible.*

Proof. Since $p \notin L$ there is by the separation axiom a countable open cover of L, say $U_1, U_2, \ldots$ with $p \notin \mathrm{cl}\, U_i$ all i. By regularity, choose open sets $V_1, V_2, \ldots$ with $\mathrm{cl}\, V_1$ countably compact and with $p \in V_{i+1} \subseteq \mathrm{cl}\, V_{i+1} \subseteq V_i$ and $V_i \cap U_i = \varnothing$. Then $H = \bigcap V_i \in \bar{\mathcal{G}}_\delta(p)$ and $H \cap L = \varnothing$.

Definitions 3. A set D will be called *metrically separated* in M_i (abbreviated to m. s.) if D is an infinite subset of M_i and for some $n > 0$ we have $\rho_i(d, d') > 1/n$ for each pair of distinct points d, d' of D. We denote by $\nu(D)$ the least number n with this property.

We put for any set B:

$$\Sigma_{ij}^n(B) = \{x \in M_j\colon \exists \text{m.s. set } D_x \subseteq M_i \cap B \text{ with } \nu(D) = n \,\&\, x \in \text{seq-cl}(D_x)\}.$$

LEMMA 4. *If D is metrically separated in M_i then* $(\mathrm{cl}\, D \setminus D) \cap M_i = \varnothing$. $\mathrm{cl}\, \Sigma^n_{ij}(B) \cap M_i = \varnothing$.

PROOF. The first assertion is clear. Suppose that $\xi \in M_i \cap \mathrm{cl}\, \Sigma^n_{ij}(B)$ and that U is an open neighbourhood of ξ with $U \cap M_i$ refining the ball in M_i centred at ξ and of radius $1/2n$. Then for some $x \in \Sigma^n_{ij}(B)$ we have $x \in U$. Hence for two points at least of D_x, say d, d', we have $d, d' \in U \cap M_i$, so $\rho_i(d, d') < 1/n$ but $\nu(D_x) = n$.

LEMMA 5. *If $B \subseteq A$ is nonnegligible on M_i, then for some n, j, $\Sigma^n_{ij}(B)$ is a nonnegligible subset of A.*

PROOF. Suppose that for each n, j there is $H_{nj} \in \bar{\mathcal{G}}_\delta(p)$ with $H_{nj} \cap \Sigma^n_{ij}(B) = \varnothing$. Put $H = \bigcap_{nj} H_{nj} \in \bar{\mathcal{G}}_\delta(p)$. Now $H \cap B \cap M_i$ is nonnegligible, hence non-Lindelöf. Therefore for some m, $H \cap B \cap M_i$ contains a metrically separated set D with $\nu(D) = m$. Now $D \subseteq A$, so by Lemma 2, D has a point of accumulation a in $A \cap H$. Since $a \in M_j$ for some j, we have $a \in \Sigma^m_{ij}(B) \cap H$, a contradiction.

LEMMA 6. *If $\{F_n\}$ is a descending sequence of sequentially closed subsets of A and $H \in \bar{\mathcal{G}}_\delta(p)$ with $\bigcap F_n \cap H = \varnothing$, then for some n, $F_n \cap H = \varnothing$.*

PROOF. Otherwise we may choose $a_n \in F_n \cap H$ for each n. Since $\{a_n : n \in \omega\}$ has a convergent subsequence with limit $a \in A \cap H$, we see that $a \in F_n \cap H$ all n, a contradiction.

PROOF OF THEOREM 1. By Lemma 1, A is nonnegligible and so for some least i, say i_0, $A \cap X_i$ is nonnegligible (since $\bar{\mathcal{G}}_\delta(p)$ is closed under countable intersections). Choose $H_0 \in \bar{\mathcal{G}}_\delta(p)$ so that, on writing $A_0 = H_0 \cap A$,

$$A_0 \cap \bigcup_{i < i_0} X_i = \varnothing. \tag{1}$$

By Lemma 5, $\Sigma^n_{i_0 j}(A_0)$ is nonnegligible for some n and some $j \neq i_0$. So let i_1 be the least $i \neq i_0$ so that $A_0 \cap X_i$ is nonnegligible. By (1), $i_1 > i_0$. Choose $H_1 \in \bar{\mathcal{G}}_\delta(p)$ so that, writing $A_1 = H_1 \cap A_0$, we have

$$A_1 \cap \bigcup_{i < i_1} X_i = \varnothing.$$

Continuing by induction we may choose numbers $i_0 < i_1 < \cdots < i_n \cdots$ sequentially closed sets $A_0 \supseteq A_1 \supseteq \cdots$ and sets $H_n \in \bar{\mathcal{G}}_\delta(p)$ so that $A_n = H_n \cap A_{n-1}$ and $A_n \cap X_{i_n}$ is nonnegligible and

$$A_n \cap \bigcup_{i < i_n} X_i = \varnothing. \tag{2}$$

By Lemma 6, since the A_n are sequentially closed, $\bigcap_n A_n \neq \varnothing$ and this contradicts (2). Hence, after all, A must have been closed.

My original proof used transfinite induction at this last stage and I am grateful to Eric van Douwen for suggesting the use of the H_n's to avoid it.

(Without the H_n's the induction above proceeds ad transfinitum to ω_1, picking i_α as in Lemma 1.)

The following simple corollary is of interest against the context of inductively defined topologies (see [7]).

COROLLARY. *Let X be a locally compact, σ-metric Hausdorff space which is countably compact. Then X is compact.*

PROOF. Let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Since X^* satisfies the hypotheses of Theorem 1, it is sequential. Now if ∞ were not isolated there would be a sequence $\{x_n\}$ in X with $\infty = \lim x_n$. This implies that $\{x_n\}$ has no point of accumulation in X, contradicting countable compactness. Hence $\{\infty\}$ is open and so X is compact.

We now give Kunen's proof of the following

THEOREM 3 (M. E. RUDIN, K. KUNEN). *Let X be a regular space which is countably compact. Suppose that $X = \bigcup_{n\in\omega} X_n$ and each point of X_n has countable character in X_n. Then X is sequentially compact.*

PROOF. Let D be countable and suppose that D has no convergent subsequences. Assume $X = \operatorname{cl} D$. Write $D = \{d_n : n \in \omega\}$. Let $p \in X \setminus D$ and choose open sets U_n containing p so that $D \setminus \operatorname{cl} U_n$ is infinite, $U_n \cap \{d_m : m < n\} = \varnothing$ and $\operatorname{cl} U_{n+1} \subseteq U_n$. Put $K = \bigcap_n U_n$, then K is closed, countably compact, nonempty and a $\mathcal{G}_\delta$; moreover, no point of K may have character $\aleph_0$ in K. It follows from this that for each n, $K \cap X_n$ is nowhere dense in K ($x \in K \cap \operatorname{int}(\operatorname{cl} X_n) \cap X_n$ would have countable character); but by the Baire category theorem K cannot be a countable union of nowhere dense sets.

This paper was written whilst I was on a visit to the University of Wisconsin at Madison. I wish to thank Professors M. E. Rudin and K. Kunen most sincerely for many very stimulating conversations on this and other subjects and for the hospitality of their department.

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