COMMUTATIVE PERFECT QF-1 RINGS

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ABSTRACT. If R is a commutative artinian ring, then it is known that every finitely generated faithful R-module is balanced (i.e. has the double centralizer property) if and only if R is a quasi-Frobenius ring. In this note, constructing new nonbalanced modules we prove that the assumption on R to be artinian can be replaced by the weaker condition that R is perfect.

As an example which shows that the balanced condition for modules have strong influence on the structure of rings we can give a theorem of Ringel [7] and Storrer [8], [9] that commutative noetherian strong QF-1 rings are QF (i.e. quasi-Frobenius). This is a generalization of a theorem of Camillo [1] and Dickson and Fuller [3] for commutative artinian rings, which is also a generalization of Floyd's result [4] for commutative algebras over an algebraically closed field. Here a module M over a ring R with identity 1 is called balanced if the canonical ring homomorphism of R into the double centralizer of M is surjective, and following R. M. Thrall [10] R is said to be QF-1 (resp. strong QF-1) if every finitely generated faithful (resp. every faithful) R-module is balanced.

In connection with commutative QF-1 rings V. P. Camillo [2] studied rings with the principal extension property (i.e. every module homomorphism from a principal ideal into R can be extended to an endomorphism of R) and he gave a commutative semiprimary local ring with a simple socle, which satisfies the principal extension property, but is not a PF ring (i.e. a self-injective cogenerator ring). Though he did not succeed to determine whether it is QF-1 or not, the example suggests to us a question whether commutative perfect QF-1 rings are PF (equivalently QF by Osofsky [6, Theorem 3]) as similarly as in commutative noetherian strong QF-1 rings.

The purpose of this paper is to give an affirmative answer to this question. We shall prove

THEOREM. *Let R be a commutative perfect ring. Then every finitely generated faithful R-module is balanced if and only if R is a quasi-Frobenius ring.*

The proof of the Theorem uses the following lemma concerned with constructions of new nonbalanced modules.

LEMMA 1. *Let R be a commutative local ring with the essential socle and N the Jacobson radical of R. For ideals $K_j, j = 1, 2$, let us denote $\text{Ann}_R K_j = \{r \in R | K_j r = 0\}$ by L_j. If R is QF-1 and $NK_1 \subset K_2 \subsetneq K_1$, then $\text{Hom}_R(K_1/K_2, \text{Soc } R)$ is R-isomorphic to either L_2/L_1 or $L_2/L_1 \oplus R/N$.*

PROOF. Let $\overline{A}$ be the R-endomorphism ring of K_1 and denote by A the subring of $\overline{A}$ consisting of endomorphisms induced by the multiplication with elements of R on the right hand. We can identify A with R/L_1 and then

$$\text{Hom}_R(K_1/K_2, \text{Soc } R) \cap A = L_2/L_1.$$

Suppose that there exist $\rho, \sigma \in \text{Hom}_R(K_1/K_2, \text{Soc } R) \setminus L_2/L_1$ such that $\sigma \notin A + A\rho$. By [1, 19. Lemma] and our assumption Soc R is simple and essential. Hence, considering ρ and σ as elements of $\overline{A}$, both Ker ρ and Ker σ contain Soc R and

(1) $\rho^2 = \sigma^2 = \rho\sigma = \sigma\rho = 0.$

Further as R is commutative, $a\rho = \rho a$ and $a\sigma = \sigma a$ in $\overline{A}$ for all $a \in A$, and

(2) $_R A\rho \cong {}_R A\sigma \cong {}_R R/N.$

Therefore $A + A\rho + A\sigma \; (= A \oplus A\rho \oplus A\sigma)$ may be considered as a subring of $\overline{A}$.

Now, put $S = \{(k, k\rho, k\sigma) \in R \oplus R \oplus R | k \in K_1\}$ and consider a left R-module $M = (R \oplus R \oplus R)/S$. Then M is faithful, because for any $r \in R$ $(0, 0, r) \equiv 0 \mod S$ implies $r = 0$.

The endomorphisms of $_R M$ may be lifted to endomorphisms of $_R R \oplus R \oplus R$ with S as their stabilizer, thus to matrices $(r_{ij}), r_{ij} \in R$ and $i, j = 1, 2, 3$, such that

$$(kr_{11} + k\rho r_{21} + k\sigma r_{31})\rho = kr_{12} + k\rho r_{22} + k\sigma r_{32},$$

$$(kr_{11} + k\rho r_{21} + k\sigma r_{31})\sigma = kr_{13} + k\rho r_{23} + k\sigma r_{33},$$

for all $k \in K_1$. Then by (1) and (2) we have

$$r_{11} - r_{22} \equiv r_{11} - r_{33} \equiv r_{23} \equiv r_{32} \equiv 0 \mod N$$

and $r_{12} \equiv r_{13} \equiv 0 \mod L_1$.

Choose an element b of K_1 such that $b \notin \ker \rho$ and let us define an additive homomorphism Ψ' of $R \oplus R \oplus R$ into itself by $(r_1, r_2, r_3) \mapsto (br_3, 0, 0)$. Since Ψ' maps S into S and $b \notin \text{Ker } \rho$, Ψ' induces a nonzero additive endomorphism Ψ of M. Moreover by the following way we can show that Ψ commutes with all endomorphisms of $_R M$:

$$[\Psi'(r_1, r_2, r_3)](r_{ij}) = (r_3 r_{11} b, r_3 b r_{12}, r_3 b r_{13})$$

$$\equiv (r_1 r_{13} b + r_2 r_{23} b + r_3 r_{33} b, 0, 0) \mod S$$

$$= \Psi'[(r_1, r_2, r_3)(r_{ij})].$$

On the other hand, Ψ vanishes on the submodule

$$[R(0, 1, 0) + S]/S \cong R(0, 1, 0)/[R(0, 1, 0) \cap S] \cong R(0, 1, 0).$$

Thus Ψ is not induced by the multiplication with any element of R on the left

hand. This contradicts that R is QF-1. Thus

$$\text{Hom}_R(K_1/K_2, \text{Soc } R) = L_2/L_1 \oplus A\rho$$

or L_2/L_1, and the conclusion is now clear.

LEMMA 2. *Let R be a commutative local ring with the essential socle and N the Jacobson radical. For any positive integer j let us denote $\text{Ann}_R N^j = \{r \in R | N^j r = 0\}$ by $\text{Soc}^j R$. If R is QF-1 and $\text{Soc}^{i+1} R/\text{Soc}^i R \neq 0$, $i = 1, 2, \ldots, n$, then all N^i/N^{i+1} and $\text{Soc}^{i+1} R/\text{Soc}^i R$ are finitely generated R-modules.*

PROOF. Let $L_i = \text{Ann}_R \text{Soc}^i R = \{r \in R | (\text{Soc}^i R)r = 0\}$ for $i = 1, 2, \ldots, n$. We shall proceed our proof by induction and from the assumption that R/N^k, $k < n$, is artinian we shall conclude that N^k/N^{k+1} is a finitely generated R-module. Suppose N^k/N^{k+1} is not finitely generated. By Lemma 1 we have isomorphisms

(3)
$$\begin{aligned}\text{Hom}_R(N^k/N^{k+1}, \text{Soc } R) \\ \cong \text{Soc}^{k+1} R/\text{Soc}^k R \quad \text{or} \quad \text{Soc}^{k+1} R/\text{Soc}^k R \oplus R/N\end{aligned}$$

and

(4) $\quad \text{Hom}_R(\text{Soc}^{k+1} R/\text{Soc}^{k+1} R, \text{Soc } R) = L_k/L_{k+1} \quad \text{or} \quad L_k/L_{k+1} \oplus R/N.$

Since R is a local ring, R/N is isomorphic to a field F. Then N^k/N^{k+1}, $\text{Soc}^{k+1} R/\text{Soc}^k R$, L_k/L_{k+1}, $\text{Hom}_R(\text{Soc}^{k+1} R/\text{Soc}^k R, \text{Soc } R)$ and $\text{Hom}_R(N^k/N^{k+1}, \text{Soc } R)$ are considered as F-vector spaces. Further $L_k/L_{k+1} + N^k$ is a finite dimensional F-vector space, for $L_k/L_{k+1} + N^k$ is a homomorphic image of L_k/N^k and R/N^k is artinian. Hence if

$$\text{Soc}^{k+1} R/\text{Soc}^k R \cong \oplus_\Lambda F, \quad N^k/N^{k+1} \cong \oplus_\Gamma F$$

and $L_k/L_{k+1} \cong \oplus_\Delta F$, then

$$\text{Hom}_R(\text{Soc}^{k+1} R/\text{Soc}^k R, \text{Soc } R) \cong \Pi_\Lambda F, \quad \text{Hom}_R(N^k/N^{k+1}, \text{Soc } R) \approx \Pi_\Gamma F$$

and $L_{k+1} + N^k/L_{k+1} \cong N^k/L_{k+1} \cap N^k \cong \oplus_\Delta F$. Denote the cardinalities of F, Λ, Γ and Δ by c, λ, γ and δ respectively. Then it follows by (3), (4) and our assumption that γ, λ and δ are infinte, $c^\gamma = \lambda$, $c^\lambda = \delta$ and $\delta \leqslant \gamma$ (cf. [5, IV, §4, Theorem 1]). Thus we have $c^{(c^\gamma)} \leqslant \gamma$, but this is a contradiction. Consequently both N^k/N^{k+1} and $\text{Soc}^{k+1} R/\text{Soc}^k R$ are finitely generated R-modules.

PROOF OF THEOREM. Let R be a commutative perfect QF-1 ring. A commutative perfect ring is a direct product of finitely many local rings and a direct product of finitely many rings is QF-1 if and only if each component ring is QF-1. So, without loss of generality we may assume R is local. Then it is obvious that R satisfies the assumptions in Lemmas 1 and 2. Hence N/N^2 is a finitely generated R-module and by [6, Lemma 11] R is artinian. Therefore by Camillo [1, 22. Theorem] and Dickson and Fuller [3, Theorem] R is QF.

The converse is well known. This completes the proof.

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