A GENERALIZATION
OF VON NEUMANN'S INEQUALITY
TO THE COMPLEX BALL1

S. W. DRURY

ABSTRACT. A necessary and sufficient condition is found for a polynomial \( Q \) of \( J \) variables to be such that \( Q(A_1, \ldots, A_J) \) is a contraction whenever \( A_j \) \((1 < j < J)\) are commuting linear operators on complex hilbert space satisfying \( \sum_{j=1}^{J} A_j^* A_j < I \).

1. Introduction and notations. A celebrated theorem of J. von Neumann [1] asserts that, for \( A \) a linear contraction on a complex hilbert space and \( Q \) a complex polynomial, the inequality

\[
\|Q(A)\| \leq \sup\{|Q(z)|; z \in \mathbb{C}, |z| < 1\}
\]

holds. In this note we give a generalization of this inequality to an analogous situation in several variables in which the disc is replaced by the complex ball.

Let \( J \) be a fixed integer with \( J > 1 \). We shall consider \( J \)-tuples \( A = (A_1, \ldots, A_J) \) of linear operators on a complex hilbert space \( H \) satisfying

\[
[A_{j_1}, A_{j_2}] = 0 \quad (1 < j_1, j_2 < J) \quad \text{and} \quad \sum_{j=1}^{J} \|A_j h\|^2 < \|h\|^2 \quad \forall h \in H.
\]

For \( n = (n_1, \ldots, n_J) \) a multi-index (that is an element of \( \mathbb{Z}_+^J \)), we shall denote \( |n| = \sum_{j=1}^{J} n_j, n! = \prod_{j=1}^{J} (n_j!) \), \( \beta(n) = |n|!(n!)^{-1} \) the ‘multinomial coefficients’, and \( \alpha(n) = (J + |n| - 1)!(n!(J - 1)!)^{-1} \). If also \( z = (z_1, \ldots, z_J) \) and \( w = (w_1, \ldots, w_J) \) are elements of \( \mathbb{C}^J \), we shall use the notations

\[
|z|^2 = \sum_{j=1}^{J} |z_j|^2, \quad z \cdot w = \sum_{j=1}^{J} z_j w_j, \quad z^n = \prod_{j=1}^{J} z_j^n.
\]

Similar notations will be used for a \( J \)-tuple \( A \) satisfying (2). Note that \( A^n \) makes sense since the \( A_j \) commute, and for \( Q \) a complex polynomial of \( J \) variables we will denote \( Q(A) = Q(A_1, \ldots, A_J) \). Our main result is that there is an ‘archetypal’ \( J \)-tuple \( S \) satisfying (2) with the universal property described as follows.

THEOREM. Let \( A \) be a \( J \)-tuple satisfying (2) and let \( Q \) be a complex
polynomial of J-variables; then
\[ \|Q(A)\| < \|Q(S)\|. \]

We will give an alternative description of \(\|Q(S)\|\) in §3 which makes it clear that (3) reduces to (1) when \(J = 1\). Now we describe the \(J\)-tuple \(S\) explicitly. Let \(l^2(\mathbb{Z}_+, \beta)\) stand for the complex hilbert space of functions \(f: \mathbb{Z}_+ \to \mathbb{C}\) such that \(\sum|f(n)|^2\beta(n)\) is finite, and let \(S_j\) denote the ‘backwards shift’ operator
\[ S_jf(n) = f(n + e_j) \]
on this space. Here \(e_j\) is the multi-index given by \((e_j)_k = \delta_{jk}\) (Kronecker’s delta). Clearly the \(S_j\) commute. Routine calculations show that
\[ (S_j^*g)(n) = g(n - e_j)\beta(n - e_j)(\beta(n))^{-1} = g(n - e_j)n|n|^{-1}. \]
In the case \(n_j = 0\) the second and third expressions in (4) are interpreted as zero. Then \(I - S^* \cdot S\) is the projection operator on the 1-dimensional subspace of elements carried by the zero multi-index. In particular, (2) holds.

2. Proof of the Theorem. The proof given here is based on Nagy and Foiaș [4, Chapter I, §10]. It will be enough to show that (3) holds when (2) is replaced by
\[ \left[ A_{j_1}, A_{j_2} \right] = 0 \quad (1 < j_1, j_2 < J) \quad \text{and} \quad \sum_{j=1}^J \|A_jh\|^2 < r^2\|h\|^2 \quad \forall h \in H \]
for \(r\) satisfying \(0 < r < 1\). Let \(r\) be fixed in this range and suppose that (5) holds. Then the positive definite defect operator \(D\) given by \(D^*D = I - A^* \cdot A\) is invertible. Now define a new hilbert space \(\tilde{H}\) to have the same underlying space as \(H\) and the norm
\[ \|h\|_{\tilde{H}} = \|Dh\|_H. \]
We construct a ‘dilation’ on the space \(\tilde{H} = l^2(\mathbb{Z}_+, \beta; \tilde{H})\). The inclusion map \(\Theta: H \to \tilde{H}\) is given by
\[ \Theta h(n) = A^nh \quad \forall h \in H. \]
By elementary properties of the multinomial coefficients we have
\[ \sum_{|n| = k+1} \beta(n)A^{*n}A^n = \sum_{|m|=k} \beta(m)A^{*m}(A^* \cdot A)A^m; \]
also one shows by induction on \(k\) that \(\|\sum_{|n|=k} \beta(n)A^{*n}A^n\| < r^{2k}\); it follows that the sum
\[ \sum_{k=0}^\infty \sum_{|n|=k} \beta(n)A^{*n}(I - A^* \cdot A)A^n \]
converges absolutely and ‘telescopes’ to the identity operator. Thus
\[ \sum_n \beta(n)\|A^n h\|_{\tilde{H}}^2 = \sum_n \beta(n)\|DA^n h\|_{\tilde{H}}^2 = \|h\|_{\tilde{H}}^2. \]
so that $\Theta$ is indeed an isometric inclusion. Next let $\tilde{S}_j$ denote the (vector-valued) backwards shift on $\tilde{H}$ analogous to $S_j$. Then the intertwining relation
\[ \tilde{S}^n \Theta = \Theta A^n \quad \forall n \in \mathbb{Z}_+^J \]
holds. From this we may deduce that $\|Q(A)\| < \|Q(\tilde{S})\|$ for every complex polynomial $Q$ of $J$ variables. On the other hand, since $\tilde{S} = S \otimes I_H$, we have $Q(\tilde{S}) = Q(S) \otimes I_H$, and it follows that $\|Q(\tilde{S})\| < \|Q(S)\|$. Thus (3) holds as required.

3. Concluding remarks. In this section we relate $\|Q(S)\|$ to a certain multiplier norm. Let $\mathcal{K}$ denote the hilbert space of functions $\varphi$ which admit a power series representation
\[ \varphi(z) = \sum_n a(n) z^n \]
with $a(n) \in \mathbb{C}$ for $n \in \mathbb{Z}_+^J$ satisfying
\[ \|\varphi\|^2_{\mathcal{K}} = \sum_n \beta(n)^{-1} |a(n)|^2 < \infty. \]  

The norm condition (7) implies that the series (6) converges absolutely for $|z| < 1$ so that $\mathcal{K}$ may be viewed as a hilbert space of functions holomorphic in the open unit ball. Let $M_j$ denote the multiplication operator on $\mathcal{K}$ defined by $\varphi_j$ and let $M$ denote the corresponding $J$-tuple.

**Lemma.** $\|Q(M)\| = \|Q(S)\|$ for every polynomial $Q$.

It is worth noting that $M$ does not satisfy the second half of (2). The operator $Q(M)$ is, of course, just the multiplication operator on $\mathcal{K}$ defined by $Q$.

**Proof of the Lemma.** By virtue of (4) the isometric isomorphism $T: l^2(\mathbb{Z}_+^J, \beta) \to \mathcal{K}$ given by
\[ (Tg)(z) = \sum g(n) \beta(n) z^n \]
satisfies the intertwining relation $T S_j^* = M_j T$ ($1 < j < J$). Let $Q$ be given and define a new polynomial $\tilde{Q}$ by $\tilde{Q}(z) = Q(\bar{z})$. Then we have
\[ \|Q(S)\| = \|Q(S)^*\| = \|\tilde{Q}(S^*)\| = \|\tilde{Q}(M)\|. \]  

Also, since the isometric conjugation operator on $\mathcal{K}$ given by $\varphi \mapsto \bar{\varphi}$ intertwines $Q(M)$ and $\tilde{Q}(M)$ we have that $\|Q(M)\| = \|\tilde{Q}(M)\|$. Combining this with (8) we have the conclusion of the Lemma.

**Corollary.** Let $A$ be a $J$-tuple satisfying (2) and let $Q$ be a complex polynomial. Then $\|Q(A)\| < \|Q(M)\|$.

In the case $J = 1$, $\mathcal{K}$ is just the Szegö space $H^2$ so that $\|Q(M)\| < \|Q\|_{\infty}$ (actually equality occurs). Thus the Corollary yields the original von Neumann inequality (1).

It is well known that for large $J$ the direct generalization of von Neumann's inequality fails [2], [3]. In our context it fails for $J > 2$. Indeed in the
essentially typical case $J = 2$ we consider $Q_n(z) = (2z_1z_2)^n$. Then we have that $\|Q_n\|_\infty = \sup\{|Q_n(z)|; |z| < 1\} = 1$ whilst, since $\|1\|_\infty = 1$,
\[
\|Q_n (M)\|^2 \geq \|Q_n\|^2_\infty = 4^n(n!)^2(2n!)^{-1},
\]
which is unbounded.

Finally, we give a crude method for estimating the multiplier norm of $\mathcal{K}$. Let $\partial B_j$ denote the unit sphere in $C^j$ and $\sigma$ the normalized area measure. Let $\mathcal{L}$ denote the Hilbert space of functions $f(\lambda) = \sum_{k=0}^\infty a_k \lambda^k$ on the multiplicative circle group $T$ satisfying
\[
\sum_{k=0}^\infty |a_k|^2 \left( \frac{J+k-1}{k} \right) < \infty.
\]

For $\lambda \in T$ and $z \in C^j$ we denote by $\lambda z$ the element $(\lambda z_1, \ldots, \lambda z_j)$.

**Proposition.** Let $Q$ be a polynomial of $J$ variables, and for $z \in \partial B_j$ define a polynomial $Q_z$ of one variable by $Q_z(\lambda) = Q(\lambda z)$. Then
\[
(9) \quad \|Q (M)\| < \sup_{z \in \partial B_j} \|Q_z\|_{\text{mult}(\mathcal{L})}.
\]

Here $\|P\|_{\text{mult}(\mathcal{L})}$ denotes the norm of $P$ as a multiplier of the space $\mathcal{L}$.

The significance of the Proposition is that the right-hand side of (9) takes account of the smoothness of $Q$ on $\partial B_j$ only in the direction of “$i$ times the normal”.

**Proof of the Proposition.** It is well known [5] that the functions of the Szegö space $H^2(\partial B_j, \sigma)$ are those that admit a power series expansion $\theta(z) = \sum a(n) z^n$, where $\|\theta\|^2 = \sum (a(n))^{-1} |a(n)|^2 < \infty$. The mapping $N: \mathcal{K} \to H^2(\partial B_j, \sigma; \mathcal{L})$, defined by
\[
(N \mathcal{K})(z, \lambda) = \mathcal{K}(\lambda z) \quad \forall \mathcal{K} \in \mathcal{K}, \lambda \in T, |z| < 1,
\]
is an isometry, for if $\mathcal{K}(z) = \sum a(n) z^n$, then $N \mathcal{K}(z, \lambda) = \sum a(n) z^n \lambda^{|n|}$, and we have that
\[
\|N \mathcal{K}\|^2 = \sum_n (a(n))^{-1} \left( \frac{J+|n|-1}{|n|} \right) |a(n)|^2
= \sum_n (\beta(n))^{-1} |a(n)|^2 = \|\mathcal{K}\|^2_{\mathcal{X}}.
\]

Now let $\mathcal{X}$ be an element of $\mathcal{K}$ and assume that the right-hand side of (9) is bounded by $1$. Then we have that
\[
(N (Q \mathcal{X}))(z, \lambda) = Q_z(\lambda) (N \mathcal{X})(z, \lambda)
\]
and that
\[
(10) \quad \|(N (Q \mathcal{X}))(z, \cdot)\|_\mathcal{L} < \|(N \mathcal{X})(z, \cdot)\|_\mathcal{L}.
\]

Squaring (10), integrating with respect to the variable $z$ and using the fact that $N$ is an isometry now yields $\|Q \mathcal{X}\|_\mathcal{X} < \|\mathcal{X}\|_\mathcal{X}$. Since $\mathcal{X}$ was an arbitrary element of $\mathcal{K}$ this yields the desired result.
References


Department of Mathematics, McGill University, Montreal H3A 2K6, Canada