WEAK CONVERGENCE TO THE FIXED POINT OF AN ASYMPTOTICALLY NONEXPANSIVE MAP

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Abstract. It is proved that, in certain Banach spaces, any asymptotically nonexpansive and asymptotically regular map has the property that its iterates, applied to any point in the domain, give a sequence converging weakly to a fixed point.

Our object is to extend Opial's convergence theorem (Theorem 2 in [7]) to the case of asymptotically nonexpansive map. Suppose $K$ is a nonempty bounded closed convex subset of a Banach space $X$. A mapping $T: K \rightarrow K$ is called asymptotically nonexpansive [5] if for each $x, y \in K$

\[ \|Tx - Ty\| \leq k_i \|x - y\|, \quad i = 1, 2, \ldots, \]

where $\{k_i\}$ is a fixed sequence of positive reals such that $k_i \rightarrow 1$ as $i \rightarrow \infty$. Existence of fixed points of such a mapping when $X$ is uniformly convex has been proved by Goebel and Kirk in [5].

In §1, we recall some basic definitions and known results. In §2, after Kirk we construct what we call the asymptotically central set of a sequence and observe some simple facts about it. Our main results are contained in §3.

1. A mapping $T: K \rightarrow K$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y$ in $K$. It is called asymptotically nonexpansive if it satisfies (*) above. $T$ is asymptotically regular if for any $x \in K$, $T^n x - T^{n+1} x \to 0$, as $n \to \infty$. It is demiclosed if for any sequence $\{x_n\} \subset K$, $x_n \to x_0$ and $Tx_n \rightharpoonup y_0 \Rightarrow Tx_0 = y_0$ where $\rightharpoonup$ denotes weak convergence. The modulus of convexity of $X$ is a function $\delta: [0, 2] \rightarrow [0, 1]$ defined by $\delta(\varepsilon) = \inf(1 - \frac{1}{2}\|x + y\|: \|x\| < 1, \|y\| < 1, \|x - y\| > \varepsilon)$. It is known that $\delta$ is a nondecreasing function and is continuous on $[0, 2)$. It is also known [8], [9] that

\[ \|x\| \leq d, \quad \|y\| \leq d, \]

\[ \|x - y\| > \varepsilon \Rightarrow \frac{1}{2}\|x + y\| \leq \left(1 - \delta\left(\frac{\varepsilon}{d}\right)\right)d. \]

Opial [7] has shown that in a uniformly convex Banach space having weakly continuous duality mapping (or in a Hilbert space) if a sequence $\{x_n\}$ converges weakly to an $x_0$ then

\[ \lim_{n} \inf \|x_n - x_0\| < \lim_{n} \inf \|x_n - x\| \quad \forall x \neq x_0. \]

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It is easy to observe (see [2]) that the above inequality can be given an equivalent form in terms of \( \limsup \):

\[
(0') \quad \limsup_{n} \|x_n - x_0\| < \limsup_{n} \|x_n - x\| \quad \forall x \neq x_0.
\]

2. Let \( K \) be a nonempty bounded closed convex subset of a reflexive Banach space \( X \) and let \( \{x_n\} \) be any sequence in \( K \). Following Kirk [6] and Edelstein [4] let us define the following:

\[
r(x) = \limsup_{n} \|x_n - x\|, \quad x \in X.
\]

This \( r \) is a continuous function of \( X \) into the reals (see Edelstein [3]).

\[
\rho = \rho_K(\{x_n\}) = \inf \{ r(x) : x \in K \},
\]

\[
C_0 = \{ x \in K : r(x) = \rho \}.
\]

\( \rho \) is called the asymptotic radius of \( \{x_n\} \) in \( K \) and we prefer to call \( C_0 \) the asymptotically central set of \( \{x_n\} \) in \( K \). \( C_0 \) is a singleton if the space is uniformly convex (Proposition 2 below). In that case it is called asymptotic center.

Let \( B_n(r) \) denote the closed ball of radius \( r \) centered at \( x_n \) and define

\[
C_\epsilon = \bigcup_{i \geq 1} \left( \bigcap_{n > i} B_n(\rho + \epsilon) \right).
\]

**Proposition 1** [5]. \( C_0 = \cap_{\epsilon>0}(K \cap \overline{C_\epsilon}) \) and is a nonempty closed convex subset of \( K \).

**Proposition 2** [3]. If the space is uniformly convex then \( C_0 \) is a singleton.

The following lemma easily follows from Proposition 2 and the inequality \((0')\).

**Lemma 1.** Let \( K \) be a nonempty bounded closed convex subset of a uniformly convex Banach space having weakly continuous duality mapping. If a sequence \( \{x_n\} \subseteq K \) converges weakly to a point \( x_0 \) then \( x_0 \) is the asymptotic center of \( \{x_n\} \) in \( K \).

In the proof of Theorem 2 in [1] we have observed that the space being uniformly convex if \( T : K \to K \) is asymptotically nonexpansive, then the asymptotic center of \( \{T^n x\} \) in \( K \) for any \( x \in K \) is a fixed point of \( T \). We now prove:

**3. Lemma 2.** Let \( K \) and \( X \) be as in Lemma 1, \( T : K \to K \) an asymptotically nonexpansive mapping. Suppose \( x_0 \) is the asymptotic center of the sequence \( \{T^n x\} \) for some \( x \in K \). If the weak limit \( \xi_0 \) of a subsequence \( \{T^n x\} \) is a fixed point of \( T \), then it must coincide with \( x_0 \) (which is a fixed point as remarked above).

**Proof.** Let \( \rho \) and \( \rho' \) be the asymptotic radii respectively of \( \{T^n x\} \) and \( \{T^n x\} \). Clearly \( \rho' < \rho \). Since \( \{T^n x\} \) converges weakly to \( \xi_0 \), by Lemma 1, \( \xi_0 \)
must be the asymptotic center of \( \{ T^n x \} \) in \( K \), so that given any \( \epsilon > 0 \) we can choose an integer \( i_0 \) such that

\[
\| \xi_0 - T^{*i_0} x \| < \rho' + \epsilon/2.
\]

Since \( \xi_0 \) is a fixed point of \( T \) and \( T \) is asymptotically nonexpansive we can choose an integer \( J \) such that

\[
\| \xi_0 - T^{*j} x \| < k_j (\rho' + \epsilon/2) < \rho' + \epsilon < \rho + \epsilon \quad \text{for all } j > J.
\]

It follows therefore that \( \limsup \| \xi_0 - T^n x \| = \rho \) and \( x_0 \) being the unique point with this property we have \( \xi_0 = x_0 \). The proof is complete.

Now we prove our convergence theorem.

**Theorem.** Let \( X \) be a uniformly convex Banach space having weakly continuous duality mapping and \( K \) a nonempty bounded closed convex subset of \( X \). Suppose \( T: K \rightarrow K \) is asymptotically nonexpansive and asymptotically regular. Then for any \( x \in K \), the sequence \( \{ T^n x \} \) converges weakly to a fixed point of \( T \).

**Proof.** We will show that the asymptotic regularity of \( T \) makes every weak cluster point of \( \{ T^n x \} \) a fixed point of \( T \). In view of the above lemma this would mean that all the weak cluster points of \( \{ T^n x \} \) coincide with the asymptotic center \( x_0 \) of \( \{ T^n x \} \) in \( K \) (which is a fixed point) and would complete the proof.

Let us suppose that the subsequence \( \{ T^n x \} \) converges weakly to \( \xi_0 \). Then by Lemma 1, \( \xi_0 \) will be the asymptotic center of \( \{ T^n x \} \) in \( K \), let the asymptotic radius be \( \rho \). By asymptotic regularity of \( T \), \( (I - T) T^n x \rightarrow 0 \) as \( i \rightarrow \infty \). Since \( \{ T^n x \} \) converges weakly to \( \xi_0 \), this implies \( T^{n+1} x \) converges weakly to \( \xi_0 \). It follows in the same way that for any integer \( r \) the sequence \( \{ T^{n+1} x \} \) converges weakly to \( \xi_0 \) and thus all these sequences have the same asymptotic center \( \xi_0 \) in \( K \). We now claim that all these sequences have the same asymptotic radius \( \rho \).

We have

\[
\| \xi_0 - T^{n+1} x \| - \| \xi_0 - T^n x \| \leq \| \xi_0 - T^{n+1} x \| - \| \xi_0 - T^n x \| \rightarrow 0 \quad \text{as } i \rightarrow \infty
\]

by asymptotic regularity of \( T \). Thus

\[
\limsup \| \xi_0 - T^{n+1} x \| = \limsup \| \xi_0 - T^n x \| = \rho
\]

and our claim follows.

We now prove that \( \xi_0 \) is a fixed point of \( T \). If we can show that \( T^j \xi_0 \rightarrow \xi_0 \) as \( j \rightarrow \infty \), by continuity of \( T \) this will mean \( \xi_0 \) is a fixed point of \( T \), so let us suppose \( T^j \xi_0 \) does not converge to \( \xi_0 \). Then there is a \( d > 0 \) and a sequence \( \{ j_m \} \) of integers such that

\[
\| \xi_0 - T^{j_m} \xi_0 \| > d \quad \text{for all } m.
\]

By uniform convexity of the space, we may choose an \( \epsilon > 0 \) such that
(\rho + \epsilon)[1 - \delta (d/(\rho + \epsilon))] < \rho. Since all the sequences
\{T^n x_k\}_{i=1}^{\infty}, \quad r = 0, 1, 2, \ldots,
have the same asymptotic center \xi_0 and the same asymptotic radius \rho, there
exist integers I = I(r) such that
\[\|\xi_0 - T^n x_k\| < \rho + \epsilon/2 \quad \text{for all } i > I(r).\]
We have for any m,
\[\|T^j \xi_0 - T^n x_k\| < k_j \|\xi_0 - T^n x_k\| < k_j (\rho + \epsilon/2) \quad \text{for } i > I(0).\]
We choose an integer M such that (as \kappa_j \to 1 as \kappa_j \to \infty) \kappa_j (\rho + \epsilon/2) < \rho + \epsilon
for all \kappa_j > M, so that we have
\[\|T^j \xi_0 - T^n x_k\| < \rho + \epsilon \quad \text{for all } i > I(0) \text{ and all } m > M\]
and from (1) we have
\[\|\xi_0 - T^n x_k\| < \rho + \epsilon \quad \text{for } i > I(j_m).\]
Since \|\xi_0 - T^j \xi_0\| > \delta, (2) and (3) yield
\[\left\|\frac{\xi_0 + T^j \xi_0}{2} - T^j (x_k)\right\| < (\rho + \epsilon)\left[1 - \delta \left(\frac{d}{\rho + \epsilon}\right)\right] < \rho\]
for all \(i \geq \max\{I(0), I(j_M)\}\). This is a contradiction in view of the fact that
the sequence \{T^n x_k\}_{i=1}^{\infty} has asymptotic radius \rho in K. The proof of the
theorem is therefore complete.

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