A COUNTEREXAMPLE IN THE FACTORIZATION OF BANACH SPACE OPERATORS

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Abstract. A counterexample is given which completes the Banach space generalization of a theorem of R. G. Douglas concerning the factorization of Hilbert space operators.

In [1] R. G. Douglas proves the equivalence of three conditions related to factoring a Hilbert space operator. In [2] Mary Embry determines all but one of the possible implications among those three conditions interpreted for operators on a Banach space. This short note gives a counterexample to show that the one remaining possible implication does not hold.

In the notation of [4] define $A$ and $B$ on $(c_0)$ by $Ae_k = 0$ for $k \neq 1$ and $Ae_1 = y = (2^{-1}, 2^{-2}, \ldots)$ and $B(x_1, x_2, \ldots) = (2^{-1}x_1, 2^{-2}x_2, \ldots)$. Recall $(c_0)' = l^1$ and $(l^1)' = l^\infty$. The straightforward proof of the next lemma is omitted.

Lemma. The dual operator $B^*$ on $l^\infty$ maps $(f_1, f_2, \ldots)$ to $(2^{-1}f_1, 2^{-2}f_2, \ldots)$ and the images of $B$ and $B^*$ are $\{(x_1, x_2, \ldots) \in (c_0) : \lim 2^n x_n = 0\}$ and $\{(h_1, h_2, \ldots) \in l^\infty : \sup |2^n h_n| < \infty\}$, respectively. The operator $A^*$ on $l^\infty$ is defined by $A^*(g_1, g_2, \ldots) = g_1h$, where $h = (2^{-1}, 2^{-2}, \ldots)$ and the images of $A$ and $A^*$ are $\text{span}\{y\}$ and $\text{span}\{h\}$, respectively.

Theorem. Condition (i) below holds but (ii) does not hold:

(i) for some $c > 0$, $\|A'f\| < c\|B'f\|$ for all $f \in l^1$;
(ii) the image of $B$ contains the image of $A$—i.e. $B(c_0) \supset A(c_0)$.

Proof. From the Lemma it follows that $A''(l^\infty) \subset B''(l^\infty)$ and $A(c_0) \not\subset B(c_0)$. Theorem 1 of [2] implies that (i) above holds.

This counterexample fills a gap in [3] showing that the converse to Theorem 1 of [3] does not hold.

References


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Received by the editors June 13, 1977.


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