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GOODMAN'S CONJECTURE AND THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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ABSTRACT. Goodman's conjecture (for a bound on the modulus of the nth coefficient of a p-valent function as a linear combination of the moduli of the first p coefficients) is considered in the special case of functions which are polynomials of univalent functions. For such functions, it is shown that Goodman's conjecture is equivalent to a set of coefficient conjectures for normalized univalent functions.

Let $\mathbf{B} = \{z: |z| < 1\}$, and consider the regular functions f: $\mathbf{B} \to \mathbf{C}$ such that

$$f(z) = b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots . \tag{1}$$

If f is one-to-one and $b_1 = 1$, then f is called a *normalized univalent function*. Let S denote the class of all such functions. A regular function is *p-valent* if it assumes each value at most p times, and some value exactly p times. Let $M(p)$ be the class of all functions of the form (1) such that $f = P \circ \phi$, where P is a polynomial of degree q, $1 \leqslant q \leqslant p$, and $\phi \in S$. Lyzzaik [3] has shown that all close-to-convex functions of order p lie in $M(p)$. Thus many of the geometrically defined classes of p-valent functions lie in $M(p)$.

Goodman [1] conjectured that if $f(z) = b_1 z + b_2 z^2 + \cdots + b_n z^n + \cdots$ is at most p-valent, then

$$|b_n| \leqslant \sum_{k=1}^{p} \frac{2k(n+p)!}{(p-k)!\,(p+k)!\,(n-p-1)!\,(n^2-k^2)} |b_k| \tag{2}$$

for $n > p$. In particular, if $p = 2$, then

$$|b_3| \leqslant 5|b_1| + 4|b_2|, \tag{3}$$

and if $p = 3$, then

$$|b_4| \leqslant 14|b_1| + 14|b_2| + 6|b_3|. \tag{4}$$

In this paper we show that for functions $f \in M(p)$, Goodman's conjecture is true if and only if a certain set of coefficient conjectures is true for the class S. Also, we show that (3) and (4) are true for functions $f \in M(p)$ for $p = 2$, $p = 3$, respectively.

For any function $\phi \in S$ and positive integers $n > p \geqslant k \geqslant 1$, let

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$$[\phi(z)]^m = \sum_{n=1}^{\infty} A_n^{(m)} z^n, \tag{5}$$

and

$$E(\phi, p, k, n) = \begin{vmatrix} A_n^{(k)} & A_n^{(k+1)} & \cdots & A_n^{(p)} \\ A_{k+1}^{(k)} & A_{k+1}^{(k+1)} & \cdots & A_{k+1}^{(p)} \\ \vdots & \vdots & & \vdots \\ A_p^{(k)} & A_p^{(k+1)} & \cdots & A_p^{(p)} \end{vmatrix}.$$

When $k = p$, $E(\phi, p, k, n) = A_n^{(p)}$. In particular, if ϕ is the Koebe function $\kappa(z) = z/(1-z)^2$, then

$$\begin{aligned} |E(\kappa, p, k, n)| &= \frac{2kn \prod_{\alpha=1}^{p} (n^2 - \alpha^2)}{(p+k)!\,(p-k)!\,(n^2 - k^2)} \\ &= \frac{2k(n+p)!}{(p+k)!\,(p-k)!\,(n-p-1)!\,(n^2-k^2)}. \end{aligned}$$

This may be found in Goodman [**1**].

THEOREM 1. *If $f \in M(p)$, then for $n > p$,*

$$b_n = \sum_{k=1}^{p} E(\phi, p, k, n) b_k.$$

PROOF. By hypothesis $f = P \circ \phi$, so that

$$\sum_{n=1}^{\infty} b_n z^n = a_1 \phi(z) + a_2 [\phi(z)]^2 + \cdots + a_p [\phi(z)]^p, \tag{6}$$

where $P(z) = \sum_{j=1}^{n} a_j z^j$, and $\phi \in S$. For any $n \in \mathbf{N} = \{1, 2, 3, \ldots\}$, we may substitute (5) into (6) and collect terms, and obtain

$$b_n = \sum_{m=1}^{p} a_m A_n^{(m)}. \tag{7}$$

If we restrict ourselves to the values $n = 1, 2, \ldots, p$, then (7) becomes a system of p linear equations. Observe that $A_j^{(i)} = 0$ for $j < i$, and $A_j^{(j)} = 1$. This implies that the determinant of the coefficient matrix of the system is equal to 1. By Cramer's rule (see [**2**, p. 330]), for $m = 1, 2, \ldots, p$, we have

$$a_m = \begin{vmatrix} A_1^{(1)} & A_1^{(2)} & \cdots & b_1 & \cdots & A_1^{(p)} \\ A_2^{(1)} & A_2^{(2)} & \cdots & b_2 & \cdots & A_2^{(p)} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_p^{(1)} & A_p^{(2)} & \cdots & b_p & \cdots & A_p^{(p)} \end{vmatrix}, \tag{8}$$

where b_k is the km-component. For each a_m let $C_{m,k}$ be the cofactor of b_k. By expanding the determinant of each a_m according to the mth column we get

$$a_m = \sum_{k=1}^{p} (-1)^{m+k} b_k C_{m,k}.$$

For each m, $1 \leqslant m \leqslant p$, we substitute this expansion for a_m into (7). Thus

$$\begin{aligned} b_n &= \sum_{m=1}^{p} A_n^{(m)} \sum_{k=1}^{p} (-1)^{m+k} b_k C_{m,k} \\ &= \sum_{k=1}^{p} b_k \sum_{m=1}^{p} (-1)^{m+k} A_n^{(m)} C_{m,k}. \end{aligned}$$

We recognize that $\sum_{m=1}^{p}(-1)^{m+k}A_n^{(m)}C_{m,k}$ is the cofactor expansion of a determinant which reduces to $E(\phi, p, k, n)$. This is the desired result.

Note that for a fixed $\phi \in S$ there is a one-to-one correspondence between the functions $f = P \circ \phi \in M(p)$ and the set of p-tuples $(a_1, \ldots, a_p) \neq (0, \ldots, 0)$. Equations (6) and (8) show that there is a one-to-one correspondence between the p-tuples $(a_1, \ldots, a_p) \neq (0, \ldots, 0)$ and the p-tuples $(b_1, \ldots, b_p) \neq (0, \ldots, 0)$. In other words, given $\phi \in S$, $f = P \circ \phi \in M(p)$ may be uniquely determined in either of two ways: By choice of P, or by choice of the first p coefficients of f, $(b_1, \ldots, b_p)$.

THEOREM 2. *Let $n, p \in \mathbf{N}$, $n > p$. The following three conditions are equivalent.*

(a) $|E(\phi, p, k, n)| \leqslant |E(\kappa, p, k, n)|$ *for all* $\phi \in S$ *and every integer* k, $1 \leqslant k \leqslant p$.

(b) *Goodman's conjecture* (2) *is true for all functions* $f \in M(p)$.

(c) *Goodman's conjecture* (2) *is true for all functions* $f \in M(p) - M(p - 1)$.

PROOF. If (a) is true, then to show (b) is true just apply the triangle inequality to the result of Theorem 1, and substitute into this the inequalities in (a). Clearly (b) implies (c). A proof that (c) implies (a) would complete the proof of this theorem. However, we shall only prove that (b) implies (a), and indicate the minor modification necessary to show that (c) implies (a). Thus, let $\phi \in S$, $k \in \mathbf{N}$, $1 \leqslant k \leqslant p$. As noted above, we are free to choose $(b_1, \ldots, b_p) \neq (0, \ldots, 0)$. If we let $b_k = 1$ and $b_j = 0$ for $j \neq k$, then this uniquely determines a function $f(z) = \sum_{j=1}^{\infty} b_j z^j \in M(p)$. By Theorem 1, $|b_n| = |E(\phi, p, k, n)|$. Since we are assuming that Goodman's conjecture is true for the function f, we conclude that (a) is true. The reason that this does not show that (c) implies (a) is that our special choice of $(b_1, \ldots, b_p)$ might force $a_p = 0$, so that P would not be a polynomial of degree p. In this case $k \neq p$, and one could let $b_p = \delta \neq 0$, and achieve inequality (a) by letting δ approach 0.

Let $\phi(z) = z + c_2 z^2 + c_3 z^3 + \cdots \in S$. Here are some examples where the inequalities (a) of Theorem 2 obtain:

$$\begin{aligned}
|E(\phi, 2, 1, 3)| &= |c_3 - 2c_2^2| \leqslant |c_3 - c_2^2| + |c_2^2| \leqslant 5 = |E(\kappa, 2, 1, 3)|, \\
|E(\phi, 2, 2, 3)| &= |2c_2| \leqslant 4 = |E(\kappa, 2, 2, 3)|; \\
|E(\phi, 3, 1, 4)| &= |c_4 - 5c_2c_3 + 5c_2^3| \leqslant |c_4| + 5|c_2| \cdot |c_3 - c_2^2| \leqslant 14 \\
&= |E(\kappa, 3, 1, 4)|, \\
|E(\phi, 3, 2, 4)| &= |2c_3 - 5c_2^2| \leqslant 2|c_3 - c_2^2| + 3|c_2^2| \leqslant 14 = |E(\kappa, 3, 2, 4)|, \\
|E(\phi, 3, 3, 4)| &= |3c_2| \leqslant 6 = |E(\kappa, 3, 3, 4)|.
\end{aligned}$$

The first two of these show that inequality (3) is true for functions $f \in M(2)$. The last three show that (4) is true for $f \in M(3)$. In particular, the work of Lyzzaik [3] shows that (3) and (4) are true of close-to-convex, (thus) starlike, and convex functions of order $p = 2, p = 3$, respectively.

On the other hand, $E(\phi, 2, 1, 4) = c_4 - 2c_3c_2 - c_2^3$, so that $|E(\kappa, 2, 1, 4)| = 16$. However, we have only shown that $|c_4 - 2c_3c_2 - c_2^3| < 257/16$, so that we cannot assert the truth of Goodman's conjecture for the fourth coefficient of a function in $M(2)$.

It would be useful if one could show that, among p-valent functions, the modulus of the nth coefficient is only maximized by functions in $M(p)$. However, there has been negligible progress in this direction.

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