QUASITRIANGULAR MATRICES

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Abstract. It is shown that there exist quasitriangular operators which cannot be represented as quasitriangular matrices.

Introduction. A quasitriangular matrix is an infinite matrix $A = [a_{ij}]$ in which all entries below the subdiagonal are zero and the subdiagonal entries cluster at zero (i.e., $a_{ij} = 0$ for $i > j + 1$ and $\lim \inf |a_{j+1,j}| = 0$). A quasitriangular operator is an operator which can be expressed as the sum of a triangular matrix ($a_{ij} = 0$ for $i > j$) and a compact one. The relationship between quasitriangular matrices and quasitriangular operators, and the significance of studying that relationship in conjunction with the invariant subspace problem, are discussed by Halmos in [2]. It is shown in [2] that every bounded quasitriangular matrix defines a quasitriangular operator. Halmos then asks whether every cyclic quasitriangular operator has a quasitriangular matrix. It will be shown below that the answer is no. A few preliminary ideas are needed.

Let $A = \int \eta \, dE_\eta$ be a bounded selfadjoint operator defined on a separable Hilbert space $\mathcal{H}$. By Weyl’s theorem $A$ is the sum of a diagonal operator and a compact one. Hence $A$ is quasitriangular. Denote by $\mathcal{K}_a(A)$ the set of elements $x$ in $\mathcal{H}$ for which $\|E_\eta x\|^2$ is an absolutely continuous function of $\eta$. The subspace $\mathcal{K}_a(A)$ reduces $A$ [1, p. 104] and the restriction of $A$ to $\mathcal{K}_a(A)$ is called the absolutely continuous part of $A$. A result due to Kato [3] and Rosenblum [4] asserts that the absolutely continuous part of the operator $A$ remains stable under a trace class perturbation. In particular, if $C$ is selfadjoint and of trace class, and if $B = A + C$, then the absolutely continuous parts of $A$ and $B$ are unitarily equivalent.

Main result. The main result to be established is as follows.

Proposition. A selfadjoint operator with a nontrivial absolutely continuous part cannot be represented as a quasitriangular matrix.

Proof. Let $A$ be a selfadjoint operator with a nontrivial absolutely continuous part. Suppose that with respect to some orthonormal basis, $A$ can be represented as a quasitriangular matrix. Clearly this matrix takes the form

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\[
\begin{bmatrix}
b_1 & a_1 & 0 & 0 & 0 & \cdots \\
a_1 & b_2 & a_2 & 0 & 0 & \cdots \\
0 & a_2 & b_3 & a_3 & 0 & \cdots \\
0 & 0 & a_3 & b_4 & a_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

with some subsequence of \(\{a_n\}\) converging to zero. Furthermore, the subsequence \(\{a_{n_k}\}\) can be chosen so that \(\sum |a_{n_k}| < \infty\).

Let \(B\) be the matrix obtained from \(A\) by replacing each \(a_{n_k}\) by zero. Then \(B\) has finite dimensional invariant subspaces. In fact, \(B\) has a pure point spectrum.

If \(C = A - B\) then \(C\) is the real part of a weighted shift with weight sequence \(\{c_n\}\) satisfying \(\sum |c_n| < \infty\). Hence \(C\) is of trace class. Since \(A\) has an absolutely continuous part it follows, from the Kato-Rosenblum theorem, that \(A - C = B\) has an absolutely continuous part. But this contradicts the fact that \(B\) has a pure point spectrum.

**Corollary.** The real part of the unilateral shift does not have a quasitriangular matrix.

**References**


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