ON THE STABLE COHOMOTOPY OF $RP^\infty$

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ABSTRACT. There is a conjecture of G. B. Segal concerning the relation between the Burnside ring of $G$ and the stable cohomotopy of $BG$. When $G = \mathbb{Z}/2$ this conjecture is shown to be equivalent to the triviality of the group of homotopy classes of $RP^\infty$ into the “cokernel of $J$”.

1.1. Introduction. The Barratt-Priddy-Quillen theorem gives rise to a well-known homomorphism from the Burnside ring of a finite group, $\Omega(G)$, to the stable cohomotopy of its classifying space, $\pi_*(BG)$. Details of $\Omega(G)$ may be found in [3]. This homomorphism extends to the $I\Omega(G)$-adic completion of $\Omega(G)$ to give

$$\phi(G): \Omega(G)^\sim \rightarrow \pi_*(BG).$$

Around 1970 Graeme Segal conjectured, by analogy with $K$-theory [2], that $\phi(G)$ was an isomorphism. In this note I will relate (Theorem 1.6) Segal’s conjecture when $G = \mathbb{Z}/2$ to the following conjecture concerning $[RP^\infty, \text{Cok } J]$, the set of homotopy classes of maps from $RP^\infty$ to the “cokernel of $J$”.

1.2. Conjecture. $[RP^\infty, \text{Cok } J] = 0$.

In Remark 2.1 I will outline evidence in favour of the conjecture and give an equivalent reformulation (Conjecture 2.2). Firstly we must recall a few facts about the “cokernel of $J$”, Cok $J$. Henceforth all spaces will be 2-localised. Let $QS^\sim = \text{ind lim } \Omega^m S^m$, let $QmS^\sim$ be the “degree $m$” component and set $SG = QS^\sim$.

1.3. Cok $J$. From the solution of the Adams conjecture [6], [8] there is a commutative diagram

$$\begin{array}{ccc}
JO & \xymatrix{\rightarrow} & BSO \\
\beta \downarrow & & \alpha \downarrow \\
SG & \xymatrix{\rightarrow} & G/O \\
f \downarrow & & \downarrow e \\
JO & \xymatrix{\rightarrow} & BSO
\end{array}$$

The composites $e \circ \alpha$ and $f \circ \beta$ may be arranged to be $H$-space equivalences. The maps $e$ and $f$ are $H$-maps [5]. The common fibre of $e$ and $f$ is the $H$-space

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Cok \( J \). Diagram (1.4) induces splittings
\[
(1.5) \quad SG = JO \times \text{Cok} \ J \quad \text{and} \quad G/O = BSO \times \text{Cok} \ J
\]
satisfying \( \pi = \pi' \times 1 \).

1.6. **Theorem.** \( \phi(Z/2) \) is an isomorphism if and only if Conjecture 1.2 is true.

**Proof.** It is well known that \( \Omega(Z/2) \approx Z \oplus \hat{Z}_2 \approx K^*(BZ/2) \) [2]. If \( d: \pi_S^*(BZ/2) \to K^*(BZ/2) \) is the \( d \)-invariant for unitary \( K \)-theory it is easy to see that \( d \circ \phi(Z/2) \) is an isomorphism. Hence if \( [RP^\infty, \text{Cok} \ J] = 0 \) then
\[
\pi_S^*(BZ/2) = \left[ BZ/2, QS^* \right] \quad \text{(unbased homotopy classes of maps)}
\]
\[
\cong Z \oplus \left[ RP^\infty, SG \right]
\]
\[
\cong Z \oplus \left[ RP^\infty, JO \right]
\]
\[
\cong Z \oplus KO^* \left( RP^\infty \right) \quad \text{(easily deduced from (1.4))}
\]
\[
\cong K^* \left( RP^\infty \right) \quad \text{(see [2])}
\]
\[
\cong \Omega(Z/2)^+.
\]

Conversely if \( 0 \neq h: RP^\infty \to \text{Cok} \ J \) then composition of \( h \) with the inclusion of \( \text{Cok} \ J \) into \( SG \) gives a nonzero element \( x \in \pi_S^*(BZ/2) \). However \( d(x) = 0 \) since \( K^*(\text{Cok} \ J) = 0 \) [4], [7, 9.9].

2.1. **Remark.** It is difficult to construct elements in \( [RP^\infty, \text{Cok} \ J] \). My leading candidate for nontriviality is a composition of the form
\[
RP^\infty \xrightarrow{\Delta} BO \times BO \xrightarrow{\delta} G/O \to \text{Cok} \ J
\]
where \( \Delta \) is the diagonal, \( \delta \) is the deviation from additivity of a solution of the Adams conjecture, \( BO \to G/O \), and the third map is the projection obtained from (1.5). At one time I falsely claimed that this map was detected on the bottom dimensional \( S^6 \) in \( \text{Cok} \ J \). Ib Madsen pointed out the error. Subsequent correspondence and computation left us convinced that the above map is not detected by its induced homomorphism on \( H_*(\text{Cok} \ J) \) in any dimension less than or equal to eight. Several others familiar with \( SG \) and \( G/O \) were at first confident that they could produce a nonzero element. However, after much industry, this body of opinion is now unanimous in its belief of Conjecture 1.2.

There is other evidence. The set of stable homotopy classes \( \{ RP^\infty, RP^\infty \} \) is a 2-adic local ring. This follows from [9] and from the fact that such an \( S \)-map is a stable equivalence if and only if it induces the identity on \( H_1(RP^\infty, Z/2) \). By the Kahn-Priddy theorem [1] there is an epimorphism
\[
\{ RP^\infty, RP^\infty \} \to \pi_5^*(RP^\infty)
\]
so that the following conjecture implies conjecture 1.2.

2.2. **Conjecture.** \( \{ RP^\infty, RP^\infty \} \cong \hat{Z}_2 \), the 2-adics, generated by the class of the identity map.
REFERENCES


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