Ringing Extensions and Essential Monomorphisms

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Abstract. We study pairs of rings $R \subseteq S$ such that $\text{Hom}_{R}(S, -): R\text{-Mod} \to S\text{-Mod}$ preserves essential monomorphisms. We obtain a complete characterization of such a pair in case $S$ is a torsion-free algebra over a Noetherian domain $R \neq \text{Quot}(R)$; $S$ is then a left ideally finite $R$-algebra. The rings $R$ such that every ring extension $R \subseteq S$ satisfies the above condition are subdirect sums of certain Artinian rings. Furthermore, we study a generalization of trivial ring extensions and show that the center of a semi-Artinian ring is again semi-Artinian.

Let $R \subseteq S$ be rings with $1_S \in R$. This paper is concerned with the question when the functor $\text{Hom}_{R}(S, -): R\text{-Mod} \to S\text{-Mod}$ preserves essential monomorphisms or, equivalently, injective envelopes (condition (E)). D. Eisenbud has shown [1] that this is the case if $S$ is finitely generated as an $R$-module by elements which centralize $R$, and an argument by Formanek and Jategaonkar [2] shows that the condition $Rs = sR$ for the generating elements $s$ is sufficient; furthermore, the condition that $R S$ be finitely generated can be weakened (Example 2). Another type of example is furnished by the construction $S = R \times M$, where $M$ is an $R$-bimodule and multiplication is defined by an "associative" mapping $M \otimes_{R} M \to R$; Example 7 generalizes this situation. Our main result is Theorem 3: If $R$ is a Noetherian domain but not a field and $S$ a torsion-free $R$-algebra, then $R \subseteq S$ satisfies condition (E) if and only if $S$ is an ideally finite $R$-algebra, i.e. every nonzero left $S$-module contains a nonzero submodule which is finitely generated over $R$. The proof of the theorem is essentially set-theoretical and the nontrivial part of it is false for non-Noetherian domains $R$. In Proposition 8 we give a short proof of the reflexion of relative injectivity and its consequence for chain conditions for (E)-extensions. We close with two examples involving semi-Artinian rings (Theorems 9, 10).

Finally, let us mention the corresponding problem defined by an $R$-module $M$ and its endomorphism ring $E$. It is easy to see that, if $M$ generates all of its submodules, then $\text{Hom}_{R}(M, -): R\text{-Mod} \to E\text{-Mod}$ preserves essential monomorphisms. Under certain assumptions on $M$ the converse to this fact has been shown in [8, Theorem 2.4]. Our case, however, where we consider rings $R \subseteq S$ (not $\text{End}(R S)$) requires other methods since the multiplicative

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structure of $S$ plays an important role besides its $R$-module structure.

All rings, modules, etc. are assumed to be unitary. For a subgroup $U$ of some $S$-module $M$, denote by $(U: S)$ the largest $S$-submodule of $M$ contained in $U$.

**Proposition 1.** For any pair of rings $R \subset S$ the following conditions are equivalent.

(E) The functor $\text{Hom}_R(S, -): R\text{-Mod} \to S\text{-Mod}$ preserves essential monomorphisms.

(E') For every essential $R$-submodule $U$ of a nonzero left $S$-module $M$, $(U: S)$ is not zero (resp. essential in $S M$).

**Proof.** (E) $\Rightarrow$ (E'). Consider $M$ as an $S$-submodule of $\text{Hom}_R(S, M)$. Then $(U: S) = M \cap \text{Hom}_R(S, U)$ is essential in $M$. (E') $\Rightarrow$ (E). Since $(U: S) \cap N = (U \cap N: S)$ for every $S$-submodule $N$ of $M$, both versions of (E') are equivalent. Now let $X$ be essential in $R Y$ and denote by $U$ its inverse image under the mapping $\text{Hom}_R(S, Y) \to Y$ defined by $f \mapsto f(1)$. Then $(U: S) = \text{Hom}_R(S, X)$ which proves (E). □

Let us call a pair of rings $R \subset S$ an (E)-extension if it satisfies the above conditions. Then, for any intermediate ring $S'$ between $R$ and $S$, $R \subset S'$ is also an (E)-extension.

A family $(J_\alpha)_{\alpha<\tau}$ of submodules of some module, indexed by an ordinal $\tau$, is called a continuous chain, if it is increasing and if $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ for limit ordinals $\lambda$.

**Example 2.** Let $R \subset S$ be rings. Assume that $S S_R$ contains a continuous chain of submodules $(J_\alpha)_{\alpha<\tau}$ terminating at $J_\tau = S$ such that, for all $\alpha < \tau$, $J_{\alpha+1}/J_\alpha$ is a sum of $S$-submodules which, as left $R$-modules, are generated by finitely many elements $x_i$ satisfying $Rx_i = x_i R$. Then $R \subset S$ is an (E)-extension.

**Proof.** Let $U$ be an essential $R$-submodule of some $S M \neq 0 (0)$. We may assume that the least ordinal $\alpha$ such that $J_\alpha M \neq (0)$ is 1. Then $J_1$ contains a left $S$-ideal $L = R s_1 + \cdots + R s_i$ such that $LM \neq (0)$ and $Rs_i = s_i R$ for all $i$. Say $s_1 x \neq 0$ for some $x$ in $M$. As in the proof of [2, Theorem 4], by induction on $j < n$, we can find elements $r_j \in R$ such that $(0) \neq (Rs_1 + \cdots + Rs_j)r_j x \subset U$. Hence $(0) \neq L r_n x \subset (U: S)$. □

Our main result is the converse to Example 2 in the following special case.

**Theorem 3.** Let $R$ be a Noetherian domain different from its quotient field and $A$ an $R$-algebra which is a torsion-free $R$-module. Then the following are equivalent.

(a) $R \subset A$ is an (E)-extension. (b) $A$ contains a continuous chain of ideals $(J_\alpha)_{\alpha<\tau}$ terminating at $J_\tau = A$ such that each quotient $J_{\alpha+1}/J_\alpha$ is the sum of its left $A$-submodules which are finitely generated over $R$.

The implication (a) $\Rightarrow$ (b) will follow from the more general Theorem 3' below. For a domain $R \neq \text{Quot}(R)$ let us denote by $i(R)$ the greatest cardinal
such that every set of less than \( i(R) \) nonzero ideals of \( R \) has nonzero intersection. Then \( \aleph_0 < i(R) < \text{card}(R) \), and \( i(R) = \aleph_0 \) when \( R \) is Noetherian. (See [4] for a domain \( R \) satisfying \( i(R) = \aleph_1 \).)

**Theorem 3'.** Let \( R \) be a domain different from its quotient field and \( a \) a cardinal > \( i(R) \). Further, let \( M \) and \( N \) be torsion-free \( R \)-modules and \( (h_{a})_{a \in a} \) a linearly independent family in \( \text{Hom}_R(M, N) \) such that every essential submodule of \( N \) contains the image of some nonzero mapping in \( H = \sum_{a} Rh_{a} \). Then there exists a nonzero \( h \in H \) of rank(\( h \)) < \( a \).

We need two lemmas for the proof. Denote by \( a \) an arbitrary infinite cardinal.

**Lemma 4.** Let \( \{f_{a}/a \in a\} \subset \text{Hom}_K(V, W) \) be a linearly independent family of homomorphisms of vector spaces over some field \( K \) such that every nonzero \( f \in \sum_{a} Kf_{a} \) has rank(\( f \)) > \( a \). Then there exists a family \( \{X_{a}/a \in a\} \) of subspaces of \( V \) satisfying the following conditions.

(a) The sum \( \sum_{a} X_{a} \) is direct, (b) \( X_{a} \cap \text{Ker}(f_{a}) = (0) \) for all \( a \). (c) \( \text{dim}_{K}(\bigcap X_{a}) = a \) for every finite subset \( \{a_{i}\} \subset a \).

**Proof.** We need the following fact from linear algebra. If finitely many \( g_{1}, \ldots, g_{m} \in \text{Hom}_{K}(U, W) \) are such that the vectors \( g_{1}(u), \ldots, g_{m}(u) \) are linearly dependent for every \( u \in U \), then there exist \( k_{1} \in K \), not all zero, such that \( k_{1}g_{1} + \cdots + k_{m}g_{m} \) has finite rank. Let us note the following consequence. If every nonzero linear combination of a linearly independent set \( g_{1}, \ldots, g_{m} \in \text{Hom}_{K}(V, W) \) is of rank > \( a \), then every subspace \( U \) of \( V \) of codimension < \( a \) contains a vector \( u \) such that \( g_{1}(u), \ldots, g_{m}(u) \) are linearly independent.

We are now in a position to construct the spaces \( X_{a} \). Denote by \( \{F_{a}/a \in a\} \) the family of all finite nonempty subsets of \( a \). By transfinite induction we can define a family \( \{x_{a}/a \in a\} \subset V \) such that each family \( \{f_{a}(x_{a})/a \in F_{a}\} \) is linearly independent and that, denoting by \( S_{a} \) its linear span, the sum \( \sum_{a} S_{a} \) is direct. To see that this definition is possible put \( Y = \sum_{a \leq \beta} S_{a} \) for some \( \beta \in a \) and set \( W = Y \oplus C \) and \( U = \cap_{\epsilon \in F_{\beta}} f_{\epsilon}^{-1}(C) \). Since \( \text{dim}_{K}(Y) < a \), by the preceding remark, we can find \( x_{\beta} \in U \) such that \( \{f_{a}(x_{\beta})/a \in F_{\beta}\} \) is linearly independent and the sum \( Y + S_{\beta} \) is direct. Finally, let \( E_{\epsilon} = \{a \in a/\epsilon \in F_{a}\} \) and define \( X_{\epsilon} = \sum_{a \in E_{\epsilon}} Kx_{a} \) for all \( \epsilon \in a \). Properties (a), (b), (c) can now be easily checked. □

**Lemma 5.** Let \( R \neq \text{Quot}(R) \) be a domain and \( F \) a torsion-free \( R \)-module of rank(\( F \)) = \( a \) > \( i(R) \). Suppose further a set \( \Phi \) of submodules of \( F \) to be given, of cardinality < \( a \), such that every essential submodule of \( F \) contains some \( rU \) where \( U \in \Phi \) and \( r \in R - \{0\} \). Then \( \Phi \) must contain a module \( U \) of rank(\( U \)) < \( a \).

**Proof.** Let \( \Phi = \{U_{a}/a \in a\} \). By a routine reduction argument, we may restrict ourselves to the case when \( F = \bigoplus_{a} Rx_{a} \) is free on the basis \( \{x_{a}/a \in a\} \).
a) and \( U_a = \bigoplus_{\beta} R_s_{a\beta} x_{a\beta} \) with \( s_{a\beta} \in R \). Then let \( T_a = \{ \beta \in a / s_{a\beta} \neq 0 \} \) and assume, by way of contradiction, that \( \text{rank}(U_a) = \text{card}(T_a) = a \) for all \( a \in a \). By a set-theoretical argument we can find subsets \( S_a \subseteq T_a \) such that \( \text{card}(S_a) = a \) and \( S_a \cap S_{a'} = \emptyset \) for all \( a \neq a' \). Since \( i(R) < a \), there is a subset \( \{ r_{a_\alpha} / \alpha \in a \} \subseteq R - \{ 0 \} \) such that \( \bigcap_a R r_{a_\alpha} = \{ 0 \} \). Put \( r_{a\beta} = r_{a_\alpha} / \beta \) where \( j_\alpha : S_a \to a \) is some bijective mapping and set \( E_a = \bigoplus_{\beta \in S_a} R r_{a\beta} s_{a\beta} x_{a\beta} \). Then the direct sum \( E = \bigoplus_a E_a + \bigoplus_{\gamma \in \gamma} Rx_{\gamma} \) with \( \gamma = a - \bigcup_a S_a \) is essential in \( F \). Hence there exist \( \delta \in a \) and \( r \neq 0 \) such that \( rU_\delta \subseteq E \). Comparing coefficients now yields the contradiction \( r \in \bigcap_{\beta \in S_a} R r_{a\beta} = \{ 0 \} \). □

**Proof of Theorem 3'.** Suppose that \( \text{rank}(h) > a \) for every nonzero \( h \in H \). Then let \( K = \text{Quot}(R) \) and put \( V = K \otimes M, W = K \otimes N, h_a = 1 \otimes h_a \). Let \((X_a)\) be the family of subspaces of \( V \) corresponding to the \( f_a \)'s as described in Lemma 4 and set \( Y_a = M \cap X_a \) and \( F_a = h_a(Y_a) \). By Lemma 4, the submodule \( h_a(Y_a \cap \bigcup_a Y_a) \) of \( F_a \) has rank \( a \) for every finite subset \( \{ a_\alpha \} \) of \( a \). Denote by \( \Phi_a \) the set of all such submodules of \( F_a \). By Lemma 5, \( F_a \) must contain an essential submodule \( E_a \) not containing any of the modules \( rU \) with \( U \in \Phi_a \) and \( r \neq 0 \). Choose a submodule \( C \) of \( N \) such that the sum \( E = \bigoplus_a E_a + C \) is direct and essential in \( N \) and let \( h = r_1 h_{a_1} + \cdots + r_n h_{a_n} \) with nonzero \( r_i \in R \) and different \( a_i \) be such that \( h(M) \subseteq E \). Then \( h(\cap \bigcup_a Y_a) \subseteq E \cap \bigoplus_a E_a \) and, hence, \( r_i h_{a_i} (\cap \bigcup_a Y_a) \subseteq E_a \), contradicting the choice of \( E_a \). □

**Corollary 6.** Let the domain \( R \) satisfy \( i(R) = \aleph_0 \). Let \( N \) be a nonzero left module over some \( R \)-algebra \( A \) such that \( RN \) is torsion-free and \((U: A)\) is an essential \( A \)-submodule for every essential \( R \)-submodule \( U \) of \( N \). Then \( N \) contains a finitely generated \( R \)-submodule \( F \) such that \((F: A) \neq (0)\). □

**Proof.** Consider a nonzero \( A \)-submodule \( X \) of \( N \) of least \( R \)-rank and apply Theorem 3' to homomorphisms of the form \( a \mapsto ax \) for \( a \in A \) and \( x \in X \). □

**Proof of Theorem 3.** For a left \( A \)-module \( M \), denote by \( q(M) \) the sum of its submodules which are finitely generated over \( R \). Defining \( J_{a+1} / J_a = q(A / J_a) \), Corollary 6 shows that \( J_a = A \) for some ordinal \( \tau \). □

**Remarks.** (1) Following [7], let us call an algebra \( A \) over an arbitrary commutative ring \( R \) left ideally finite if it satisfies condition (b) of Theorem 3, or, equivalently, if any nonzero left \( A \)-module contains a nonzero submodule which is finitely generated over \( R \). Such an algebra is easily seen to be locally finite.

(2) For a non-Noetherian domain \( R \), Theorem 3, (a) \( \Rightarrow \) (b), is false, in general, as can be seen by taking the polynomial ring \( A = \mathbb{Z}[X] \) and its subring \( R = \mathbb{Z} + pA \) for some prime \( p \). For an essential \( R \)-submodule \( U \) of \( A \) \( M \neq (0) \), we have \( (0) \neq N + JU \subseteq (U: A) \), where \( N \) denotes the annihilator of \( J = pA \) in \( M \). But \( A \) is not integral over \( R \). □

**Example 7.** Let \( R \subseteq S \) be rings and assume that \( S_R \) contains a submodule \( J \) such that \( S = R + J \) and, given any sequence \( s_0, s_1, \ldots \), of elements of \( J \), there
is an index \( n \) such that \( s_n s_{n-1} \cdots s_0 \in R \). Then \( R \subset S \) has property (E) and every nonzero left \( S \)-module \( M \) contains a finitely generated \( R \)-submodule \( F \) such that \((F: S) \neq (0)\).

**Proof.** Suppose \((U: S) = (0)\) for an essential \( R \)-submodule \( U \) of some \( _SM \neq (0) \). Then \( Ju \not\subset U \) for every nonzero \( u \in U \). Fixing such an element \( u \) and using the fact that \( U \) is \( R \)-essential we obtain a sequence \( s_0, s_1, \ldots \) from \( J \) such that \( s_n s_{n-1} \cdots s_0 u \notin U \) for all \( n \), a contradiction. To prove the second statement assume \((F: S) = (0)\) for every f.g. \( _RF \) in \( M \) and let \( x \in M - \{0\} \). In a way similar to the preceding argument we can find elements \( s_n \in J \) and f.g. \( R \)-submodules \( F_n \) such that \( x \in F_n \) and \( s_n s_{n-1} \cdots s_0 x \notin F_n \) for all \( n \), which gives the contradiction. □

**Remark.** Let \( R \subset S \) be as above and assume that, in addition, \( J \) is an ideal in \( S \). Then, as for trivial ring extensions, every injective left \( S \)-module can be shown to be isomorphic to \( \text{Hom} _R(S, X) \) for some \( _RX \). □

The results of [1] and [2] on descent of chain condition for ring extensions of the type of Example 2 apply also to those described in Example 7. Let us mention the following footnote to Eisenbud's paper.

**Proposition 8.** Let \( R \subset S \) be rings. (a) If condition (E) holds and if \( _SM \) and \( _RX \) are such that \( \text{Hom} _R(S, X) \) is \( M \)-injective, then \( X \) is \( _RM \)-injective (condition (I)). (b) Assume that condition (I) holds and that every nonzero \( _SM \) contains some f.g. \( R \)-submodule \( F \) such that \((F: S) \neq (0)\). Then any Noetherian left \( S \)-module is Noetherian as an \( R \)-module.

**Proof.** (a) Recall a module \( X \) being \( Y \)-injective if every homomorphism \( Z \rightarrow X, \ Z \) a submodule of \( Y \), can be extended to \( Y \). Since this is the case if and only if \( \text{Hom}(Y, X) \rightarrow \text{Hom}(Y, E(X)) \) is an isomorphism for \( E(X) \) an injective hull of \( X \), the statement follows by a straightforward adjointness argument. (b) Let \( _SM \) be Noetherian. From the second part of the assumption it follows that \( _RM \) is finitely generated (consider a maximal \((F: S) \) with \( _RF \) f.g.). By a well-known argument due to Bass it suffices to show \( _RM \) to be \( X \)-injective for \( X \) an arbitrary direct sum of injective \( R \)-modules. But \( \text{Hom} _R(S, X) \) is \( M \)-injective since every submodule of \( _SM \) is f.g. over \( R \). Thus condition (I) yields the conclusion. □

In closing, let us note two examples of \( (E) \)-extensions of semi-Artinian rings. The following statement can be expressed by saying that the pair \( R \subset S \) in question has property (E).

**Theorem 9.** The center \( R \) of a left semi-Artinian ring \( S \) is semi-Artinian.

**Proof.** Let \( x \) be a central element of an arbitrary ring \( S \) with left socle \( I \). If \( Sx^n \cong Sx^{n+1} \) (mod \( I \)) for some \( n > 0 \), then \( Sx^m = Sx^{m+1} \) for some \( m > 0 \). To see this, let \( s \in S \) such that \( y = x^n - sx^{n+1} \in I \). Since \( Sx \) is Artinian, we get \( Sy^k = Sy^{k+1} \) for some \( k > 1 \). Centrality of \( x \) now yields \( x^{nk} \in Sx^{nk+1} \), so \( m = nk \) does it.

Next we claim that \( Sx^n = Sx^{n+1} \) for every \( x \in R \) and some \( n \) depending
on \( x \). Let \((I_\alpha)_{\alpha<\gamma}\) be the Loewy series of \( _SS \), i.e. the continuous chain of ideals defined by \( I_\alpha = S \) and \( I_{\alpha+1}/I_\alpha = \text{soc}(S_S/I_\alpha) \). We must show that the least ordinal \( \alpha \) such that \( Sx^n \equiv Sx^{n+1} \) (mod \( I_\alpha \)) for some \( n > 0 \) is zero. If not, \( \alpha = \beta + 1 \). Then consider the ring \( S' = S/I_\beta \), \( x' = x + I_\beta \), and \( I' = I_\alpha/I_\beta \). By the preceding remark it follows that \( S'x^m = S'x^{m+1} \) for some \( m > 0 \), contradicting the minimality of \( \alpha \). It follows that every \( x \in R \) satisfies some equation \( Rx^n = Rx^{n+1} \) [5, Satz 2.5], i.e. \( R \) has Krull dimension zero. Hence every maximal left ideal of \( S \) has maximal intersection with \( R \). Thus \( S \) is a semi-Artinian \( R \)-module. Q.E.D.

**Theorem 10.** For any ring \( R \) the following conditions are equivalent.

(a) Every ring extension \( R \leftarrow S \) has property (E).

(b) \( R/t(R) \), \( t(R) \) being the torsion part of \((R, +)\), and \( R/pR \), for every prime \( p \), are semisimple rings.

(c) \( R = A \times B \), with \( B \) a semisimple \( Q \)-algebra and \( A \) a subring of the product \( \prod_p A_p \) of Artinian rings \( A_p \) satisfying \( \text{rad}(A_p) = pA_p \) for every prime \( p \), such that \( A \) contains the ideal \( I = \bigoplus_p A_p \) and \( A/I \) is semisimple.

**Proof.** (a) \( \Rightarrow \) (b) For any homomorphism of rings \( h: R \to T \) and simple left \( T \)-module, the induced \( R \)-module \( M(h) \) is semisimple. This can be seen by making \( M \) a simple module over \( S = R \times T \) and applying Proposition 8(a) to \( R \leftarrow S \). Thus, any left \( R \)-module whose ring of endomorphisms contains a subfield must be semisimple. In particular, so are \( \mathbb{Q} \otimes Z R/t(R) \) and \( R/pR \).

(c) \( \Rightarrow \) (b) \( \Rightarrow \) (a) is easy.

(b) \( \Rightarrow \) (c) Denote by \( t_p(R) \) the \( p \)-component and by \( d(R) \) the divisible part of the additive group of \( R \). Since \( R/t(R) \) is divisible, the argument from [3, Lemma 2] yields \( R = p^n R \oplus t_p(R) \) for some \( n > 0 \). Setting \( B = d(R) \) we have \( B \cap t(R) = (0) \) since all components of \( R \) are bounded. Thus \( B \) is a module over \( R/t(R) \). These modules are easily seen to be injective over \( R \). Thus \( R = B \oplus A \) with \( A \) a left ideal. \( A \) is also a right ideal because the right annihilator of \( B \) in \( R \) has zero intersection with \( B \). Finally, consider the unitary ring \( A_p = t_p(R) \). Since it is bounded as a group and since \( A_p/pA_p \cong R/pR \) is semisimple, it is Artinian with radical \( pA_p \). The remaining part is obvious. \( \Box \)

**Remark.** Let \( A \) be an Artinian ring such that \( \text{rad}(A) = pA \) for some prime \( p \). Then \( A \) is the product of a finite number of full matrix rings over local Artinian rings \( A' \) satisfying \( \text{rad}(A') = pA' \). For more information about these rings \( A' \), see [6].

**References**


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