SHAPE TRIVIALITY AND METRIC CONTRACTIONS

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Abstract. Let $(X, d)$ be a nonempty compact metric space such that for every $\varepsilon > 0$ there exists a map $f: X \to X$ satisfying

(i) $d(x, f(x)) < \varepsilon$ for every $x \in X$, and

(ii) $d(f(x), f(y)) < d(x, y)$ for every $x, y \in X$.

Then, as proved in this paper, the shape of $X$ is trivial. This improves an earlier result of K. Borsuk [1], who proved that, under the same assumptions, $X$ is acyclic.

A function $f: X \to X$ of a metric space $(X, d)$ is called a metric contraction if

$$d(f(x), f(y)) < d(x, y) \quad \text{whenever } x, y \in X, x \neq y.$$  

Answering a question posed by Nadler, Jr. [2], Borsuk proved the following:

Theorem (Borsuk [1]). Let $(X, d)$ be a compact metric space satisfying the following condition:

(\*) for every $\varepsilon > 0$ there exists a metric contraction $f: X \to X$ such that $d(f(x), x) < \varepsilon$ for every $x \in X$.

Then $X$ is acyclic.

In this paper we shall prove a stronger result:

Theorem 1. Let $(X, d)$ be a compact metric nonempty space satisfying condition (\*). Then the shape of $X$ is trivial.

First let us observe that if $f: X \to X$ is a metric contraction of a compact metric space $X$, then $f(A) = A$ can hold for a closed subset $A$ of $X$ only if $A$ has at most one point. Thus the following theorem is more general than Theorem 1.

Theorem 2. Let $X$ be a compact Hausdorff nonempty space satisfying the following condition:

(\**) for every neighbourhood $U$ of $\Delta_X = \{(x, x): x \in X\}$ in $X \times X$ there exists a $U$-shift $f: X \to X$ such that if $f(A) = A$ for a closed subset $A$ of $X$ then $A$ has at most one point.

Then $X$ has trivial shape. ($f: X \to X$ is called a $U$-shift if $f$ is continuous and $(x, f(x)) \in U$ for every $x \in X$.)

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Proof. We have to show that arbitrary continuous map \( g: X \to P \) of \( X \) into an arbitrary finite polyhedron is homotopic to a constant map.

Let \( V \) be a neighbourhood of \( \Delta_P \) in \( P \times P \) such that continuous maps \( g_1, g_2: X \to P \) are homotopic whenever \( (g_1(x), g_2(x)) \in V \) for all \( x \in X \). Since finite polyhedron \( P \) is an ANR, such \( V \) exists. Next, let \( U = (g \times g)^{-1}(V) \), where \( g: X \to P \) is a continuous map. Consider a map \( f: X \to X \) given by (**) and the family \( F \) of all closed nonempty subsets \( A \) of \( X \) such that:

(i) \( f(A) \subseteq A \),

(ii) for every neighbourhood \( W \) of \( A \) in \( X \) there exists a map \( j: X \to X \) such that \( j(X) \subseteq W \) and \( g \circ j \) is homotopic to \( g: X \to P \).

Obviously \( X \in F \). It is also easy to see that the intersection of a chain of \( F \) belongs to \( F \). Thus, by the Kuratowski-Zorn theorem, \( F \) has a minimal member \( A_0 \).

Let \( W \) be an arbitrary neighbourhood of \( f(A_0) \) in \( X \). Let \( W_0 = f^{-1}(W) \). Then there exists \( j: X \to X \) such that \( j(X) \subseteq W_0 \) and \( g \circ j \) is homotopic to \( g \). Since

\[
(j(x), f \circ j(x)) \in U \quad \text{for every } x \in X,
\]

hence \( g \circ f \circ j \) is homotopic to \( g \circ j \), i.e. \( f \circ j \) is a map such that \( (f \circ j)(X) \subseteq W \) and \( g \circ (f \circ j) \) is homotopic to \( g \). This means that (ii) holds for \( A = f(A_0) \).

Also

\[
f(A) = f(f(A_0)) \subseteq f(A_0) = A.
\]

Thus \( f(A_0) \in F \). Since \( f(A_0) \subseteq A_0 \) and \( A_0 \) is a minimal member of \( F \), hence \( f(A_0) = A_0 \). By (**) \( A_0 = \{a\} \) is a one-point set (i.e. \( a \) is the unique fixed point of \( f \)). Let \( W_1 = \{ x \in X: (a, x) \in U \} \) and \( j: X \to X \) be such that \( j(X) \subseteq W_1 \) and \( g \circ j \) is homotopic to \( g \). But \( g \circ j: X \to P \) is homotopic also to the constant map \( x \mapsto g(a) \). The theorem is proved.

Remark. Our considerations above may be easily applied to obtain the following simple facts:

1. If \( X \) is a nonempty compact Hausdorff space and \( f: X \to X \) is a continuous map, then there exists a nonempty closed subset \( A \) of \( X \) such that \( f(A) = A \) and \( f(B) \) is not contained in \( B \) for any proper closed subset \( B \) of \( A \).

2. If \( f: X \to X \) is a metric contraction of a compact metric space \( X \) then there exists a unique \( a \in X \) such that \( f(a) = a \). Furthermore, the sequence \( y, f(y), f^2(y), \ldots \) is convergent to this unique fixed point \( a \) for every \( y \in X \).

Indeed, let \( Y \) be the set of all limit points of the sequence \( y, f(y), f^2(y), \ldots \). Then \( f(Y) = Y \). Thus \( Y = \{a\} \) and \( a = \lim_n f^n(y) \).

References


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