AN IMPLICIT FUNCTION THEOREM WITHOUT DIFFERENTIABILITY

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Abstract. We combine a "global" version of the classical inverse function theorem with Schauder's fixed point theorem to investigate the existence and continuity properties of a function \( (F, x) \rightarrow \eta(F, x) \) such that \( \eta(F, x) = F(\eta(F, x), x) \).

Let \( \mathcal{Y} \) be a Banach space, \( Y, K \subseteq \mathcal{Y}, Y \) open, \( K \) compact, \( X \) a Hausdorff space, and \( F: Y \times X \rightarrow K \) a continuous function. There are two classical theorems that ensure the existence of some set \( \bar{X} \subset X \) and of an implicit function \( \eta: \bar{X} \rightarrow Y \) such that

\[
\eta(x) = F(\eta(x), x) \quad (x \in \bar{X}),
\]

namely the implicit function theorem (IFT) and Schauder's fixed point theorem. We shall combine a "global" variant of IFT with Schauder's theorem to investigate the existence and continuity of a function \( (F, x) \rightarrow \eta(F, x) \) such that \( \eta(F, x) = F(\eta(F, x), x) \) for \( x \in X \) and for continuous \( F: Y \times X \rightarrow K \) that are sufficiently "close" to some \( H: Y \rightarrow K \) with a fixed point \( y_0 \).

This last problem arose in the study of the controllability at a control \( u_0 \) of the functional-integral equation

\[
y(t) = \int f_0(t, \tau, \xi(y)(\tau), u(\tau)) \mu(d\tau) \quad (t \in T),
\]

where \( T \) is a compact metric space, \( \mu \) a positive nonatomic Radon measure on \( T, y \in C(T, \mathbb{R}^n) \), \( u \) a control function, and \( \xi \) a "\( p \)-hereditary" [4, p. 203] transformation from \( C(T, \mathbb{R}^n) \) to \( L^\infty(\mu, \mathbb{R}^k) \). This equation, in which \( f_0 \) is Lipschitz continuous (but not necessarily differentiable) with respect to its third argument, is investigated by approximating \( f_0 \) uniformly with appropriate functions \( f_j \) which are \( C^1 \) in that argument.

Our present results are summarized in Lemma 1 and in Theorem 1 (which generalizes [5, Theorem 3.1, p. 20]). We write \( I \) for the identity mapping in \( \mathcal{Y} \), \( S(a, a) \) [\( S^c(a, a) \)] for the open [closed] ball in \( \mathcal{Y} \) of center \( a \) and radius \( a \),

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For the distance between a point $y$ and a set $A$, $d[A, y]$ for “equal by
definition”, and

$$S(A, \alpha) = \{ y \in \mathcal{O} | d[A, y] < \alpha \}.$$  

**Theorem 1.** Let $\mathcal{O}$ be a Banach space, $Y$ an open subset of $\mathcal{O}$, $K$ a compact
subset of $\mathcal{O}$, and $X$ a Hausdorff space. Suppose that $0 < \alpha, c < \infty, S^F(y_0, \alpha)$
$\subset Y, G_0: Y \rightarrow K$ is $C^1$, and

$$\left| \left[ I - G_0(y) \right]^{-1} \right| < c \quad (y \in Y).$$

Let $\tilde{K} = K \cap S^F(y_0, \alpha)$, and let $\mathcal{F}$ be the metric space of all continuous
functions $F: Y \times X \rightarrow K$ such that

$$| F(y, x) - G_0(y) + G_0(y_0) - y_0 | < \frac{\alpha}{c} \quad (y \in Y),$$

with two elements of $\mathcal{F}$ identified if they coincide on $\tilde{K} \times X$, and with the metric

$$\rho(F, F_1) = \sup \{ | F(y, x) - F_1(y, x) | \mid (y, x) \in \tilde{K} \times X \}.$$  

Then

(I) For each $(F, x) \in \mathcal{F} \times X$, the equation $y = F(y, x)$ has a solution
$y \in \tilde{K}$;

(II) There exists a Borel measurable function $\eta: \mathcal{F} \times X \rightarrow \tilde{K}$ such that

$$\eta(F, x) = F(\eta(F, x), x) \quad (F \in \mathcal{F}, x \in X);$$

(III) If $F \in \mathcal{F}$, $x \in X$ and if $\Phi(F, x) = \{ y \in \tilde{K} \mid y = F(y, x) \}$ is a singleton
at $(F, x)$ then every selection $\tilde{\eta}$ of the set-valued mapping $\Phi$ is continuous at
$(F, x)$;

(IV) If $X$ is compact, $F \in \mathcal{F}$, and $\Phi(F, x)$ is the singleton $\{ \eta(F, x) \}$ for each
$x \in X$, then $\eta(F, \cdot)$ is continuous and

$$\lim_{F \rightarrow F_0} \tilde{\eta}(F_1, x) = \eta(F, x) \text{ uniformly on } X$$

for every selection $\tilde{\eta}$ of $\Phi$.

**Remark (added in proof).** The following, more general, proposition can
be proved exactly as statement (I): Let $G: Y \rightarrow \mathcal{O}$ be $C^1$, $\left| G'(y) \right| < c$
$(y \in Y), G(y_0) = 0$, and let $\Gamma: Y \rightarrow K$ be continuous and such that $\left| \Gamma(y) \right| < \beta < \alpha/c$ $(y \in Y)$. Then the equation $G(y) + \Gamma(y) = z$ has a solution $y \in
S^F(y_0, \alpha)$ for every $z \in \mathcal{O}$ with $|z| < \alpha/c - \beta$.

In order to prove Theorem 1 we shall first require a “global” version of the
classical inverse function theorem.

**Lemma 1.** Let $Y$ be an open subset of a Banach space $\mathcal{O}$, $0 < \alpha, c < \infty, S^F(y_0, \alpha)$
$\subset Y$ and $G: Y \rightarrow \mathcal{O}$ a $C^1$ function such that

$$G(y_0) = 0, \quad \left| G'(y)^{-1} \right| < c \quad (y \in Y).$$

Then there exists a unique $C^1$ function $u: S^F(0, \alpha/c) \rightarrow S^F(y_0, \alpha)$ such that

$$G(u(x)) = x \quad (|x| < \alpha/c), \quad u(0) = y_0.$$

**Proof.** We first recall that, by the classical inverse and implicit function
theorems (as stated, e.g. in [2, pp. 265, 268]), for every point \( \eta_0 \in Y \) and a corresponding \( x_0 = G(\eta_0) \), there exist \( \varepsilon_0 > 0 \) and a unique \( C^1 \) function \( v_0: S(x_0, \varepsilon_0) \to Y \) such that
\[
G(v_0(x)) = x \quad (|x - x_0| < \varepsilon_0), \quad v_0(x_0) = \eta_0.
\]
Furthermore, if \( A \) is a connected open subset of \( \mathbb{Y} \), \( x_0 \in A \), and \( u_i: A \to Y \) and \( u_2: A \to Y \) are two continuous functions such that
\[
G(u_i(x)) = x \quad (i = 1, 2, x \in A), \quad u_1(x_0) = u_2(x_0)
\]
then \( u_1 = u_2, u_1 \) is \( C^1 \) and \( u_i(x) = G'(u_i(x))^{-1}, (x \in A) \). We shall henceforth refer to the above assertions as IFT.

Let \( a \in \mathbb{Y} \) and \( |a| = 1 \). We shall denote by \( \mathbb{B} \) the collection of all points \( \beta \in [0, \alpha/c] \) such that there exist \( \varepsilon_\beta > 0 \), a corresponding open convex set \( U_\beta = S([0, \beta]a, \varepsilon_\beta) \) and a \( C^1 \) function \( v_\beta: U_\beta \to Y \) satisfying
\[
G(v_\beta(x)) = x \quad (x \in U_\beta), \quad v_\beta(0) = y_0.
\]
It follows from IFT that \( 0 \in \mathbb{B} \) and it is clear that \( \mathbb{B} \) is a relatively open subinterval of \([0, \alpha/c]\).

Now let \( \overline{\beta} = \sup \mathbb{B} \) and \( U = \Delta \cup_{\beta \in \mathbb{B}} U_\beta \). Then \( U \) is an open and connected neighborhood of \( \cap a \) and, by IFT, \( v_\beta(x) = v_\gamma(x) \) if \( \beta, \gamma \in \mathbb{B} \) and \( x \in U_\beta \cap U_\gamma \). We may therefore define a unique \( C^1 \) function \( v: U \to Y \) by
\[
v(x) = v_\beta(x) \quad (\beta \in \mathbb{B}, x \in U_\beta).
\]
We have \( |v'(x)| = |G'(v(x))|^{-1} \leq c \) and therefore
\[
|v(\beta a) - v(0)| = |v(\beta a) - y_0| \leq c \beta \leq \alpha \quad (\beta \in \mathbb{B}).
\]
Thus \( w = \lim v(\beta a), (\beta \to \overline{\beta}, \beta \in \mathbb{B}) \) exists, \( w \in S^F(y_0, \alpha) \subset Y \) and \( G(w) = \overline{\beta} a \). Again, by IFT, there exists \( \varepsilon > 0 \) and a unique \( C^1 \) function \( \tilde{v}: S(\beta a, \varepsilon) \to Y \) satisfying
\[
G(\tilde{v}(x)) = x \quad (|x - \beta a| < \varepsilon), \quad \tilde{v}(\beta a) = w,
\]
\[
\tilde{v}(x) = v(x) \quad (\beta \in \mathbb{B}, x \in U_\beta \cap S(\beta a, \varepsilon)).
\]
Since \( \beta' = \overline{\beta} - \frac{1}{2} \varepsilon \in \mathbb{B} \), the function \( v \) is defined on \( A_{\beta'} = \Delta S([0, \beta']a, \varepsilon') \), where \( \varepsilon' = \Delta \min(\frac{1}{2} \varepsilon, \varepsilon_\beta) > 0 \). We may therefore define a \( C^1 \) function \( u^a \) by
\[
U^a = \Delta S([0, \beta]a, \varepsilon), \quad u^a(x) = v(x) \quad (x \in A_{\beta'}),
\]
which shows that \( \overline{\beta} = \sup \mathbb{B} \in \mathbb{B} \). Thus \( \mathbb{B} \) is a nonempty, open and closed subset of \([0, \alpha/c]\); hence \( \mathbb{B} = [0, \alpha/c] \).

We now conclude that for every \( a \in \mathbb{Y} \) with \( |a| = 1 \) there exist an open connected neighborhood \( U^a \) of \([0, \alpha/c]a \) and a unique continuous function \( v^a: U^a \to Y \) such that
\[
G(v^a(x)) = x \quad (x \in U^a), \quad v^a(0) = y_0.
\]
Furthermore, by IFT, \( v^a(x) = v^b(x) \) if \( x \in U^a \cap U^b \). We may therefore
define a unique continuous $u: S^F(0, \alpha/c) \to Y$ by
\[ u(x) = v^a(x) \quad (a \in \mathbb{Q}, |a| = 1, x \in U^a), \]
and it follows from IFT that $u$ is $C^1$ and $|u(x) - y_0| < c|x| < \alpha$ for all $x \in S^F(0, \alpha/c)$. Q.E.D.

**Proof of Theorem 1.** Let $H(y) = \Delta G_0(y) + y_0 - G_0(y_0)$. Then, by Lemma 1, there exists a unique $C^1$ function $u: S^F(0, \alpha/c) \to S^F(y_0, \alpha)$ such that
\[
(1) \quad (I - H)(u(v)) = v \quad (|v| \leq \alpha/c), \quad u(0) = y_0.
\]
If $F \in \mathbb{F}$ and $x \in X$ then $|F(y, x) - H(y)| \leq \alpha/c \quad [y \in S^F(y_0, \alpha)]$. Thus $y \to u(F(y, x) - H(y))$ is a continuous mapping of $S^F(y_0, \alpha)$ into itself. Furthermore, since $F(y, x) - H(y) \in K - K + G_0(y_0) - y_0$, this mapping carries $S^F(y_0, \alpha)$ into a compact set. Therefore, by Schauder's fixed point theorem, this mapping has a fixed point $\eta$. In view of (1), we have
\[
\eta = u(F(\eta, x) - H(\eta)) = H(\eta) + F(\eta, x) - H(\eta) = F(\eta, x).
\]
This proves statement (I).

Now let
\[ \Phi(F, x) = \{ y \in \tilde{K} | y = F(y, x) \} \quad (F \in \mathbb{F}, x \in X), \]
\[ \text{Graph}(\Phi) = \{ ((F, x), y) | y \in \Phi(F, x) \}. \]
Then, by statement (I), the set $\Phi(F, x)$ is nonempty for all $(F, x) \in \mathbb{F} \times X$, and it is easy to see that each $\Phi(F, x)$ is closed in the compact space $\tilde{K}$ and $\text{Graph}(\Phi)$ is closed in $(\mathbb{F} \times X) \times \tilde{K}$. It follows that

(i) the set-valued mapping $\Phi$ is Borel measurable, that is, for each closed $C \subset \tilde{K}$, the set $\{(F, x) | \Phi(F, x) \cap C \neq \emptyset \}$ is Borel measurable (and, in fact, closed); and

(ii) by a theorem of Berge [1, Corollary to Theorem 7, p. 112], for every $(F, x) \in \mathbb{F} \times X$ and every open subset $U$ of the space $\tilde{K}$, with $\Phi(F, x) \subset U$, there exists a neighborhood $V$ of $(F, x)$ in $\mathbb{F} \times X$ such that $\Phi(F, x) \subset U$ for all $(F, x) \in V$.

By a known measurable selection theorem [3, Theorem 4.1, p. 867], it follows from (i) that $\Phi$ has a Borel measurable selection $(F, x) \to \eta(F, x)$, which proves statement (II). Statement (III) follows directly from (ii), and (IV) follows from (III). Q.E.D.

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**References**


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