ON THE RELATIONS BETWEEN SOME RATE-OF-GROWTH CONDITIONS

T. G. MCLAUGHLIN

Abstract. We discuss the implications and nonimplications between four rate-of-growth properties of sets useful in certain areas of recursion theory; all nonimplications are established within the boolean algebra generated by the recursively enumerable sets.

If $S$ is an infinite set of natural numbers, we denote by $p_S$ that strictly increasing function (the so-called principal function of $S$) from the set $\mathbb{N}$ of all natural numbers into $\mathbb{N}$ whose range is $S$. In the present paper, $S$ and $T$ (with or without subscripts) always denote infinite subsets of $\mathbb{N}$. For any function $f$, $\delta f$ denotes the domain of $f$. $\Sigma^0_0$ denotes the class of all recursively enumerable subsets of $\mathbb{N}$, while $\Pi^0_0$ denotes $\{S \mid N - S \in \Sigma^0_0\}$. By a d.r.e. set, we mean one which is the difference of two elements of $\Sigma^0_0$. $S$ denotes the Turing degree of $S$, and $\overline{S}$ denotes the complement of $S$ in $\mathbb{N}$. Let $\langle \varphi_i \rangle$ be some standard recursive enumeration of the partial recursive functions of one argument. As usual, $\mu$ denotes the least number operator.

We wish to catalog the various implications and nonimplications among the universal quantifications of the following four “rate-of-growth” conditions which have been studied in [1], [2], [5], [6], and various other places in the recursion-theoretic literature; in the case of each nonimplication, we shall locate a counterexample within one or another familiar subclass of the $\exists \forall \cap \forall \exists$ level of the arithmetical hierarchy.

\[ [D_i(S)] \quad (\exists m)(\forall n > m)[n \in \delta \varphi_i \Rightarrow p_S(n) > \varphi_i(n)]; \]

\[ [D^*(S)] \quad (\exists m)(\forall n > m)[p_S(n) \in \delta \varphi_i \Rightarrow p_S(n + 1) > \varphi_i(p_S(n))]; \]

\[ [D^{**}(S)] \quad S \subseteq \delta \varphi_i \Rightarrow [D^*(S)]; \]

\[ [UH_i(S)] \quad \varphi_i \text{ total} \Rightarrow [D^*(S)]. \]

Let $D(S)$ mean $(\forall i)[D_i(S)]$; similarly for the notations “$D^*(S)$”, “$D^{**}(S)$”, “$UH(S)$”. (We have chosen the notation “$UH$” since the condition $UH(S)$ has several times been referred to in the literature as uniform hyperimmunity of $S$; in the other cases, “$D$” is for domination.)

We begin by stating a result from [1] which is just a bit weaker than one of the facts we shall need:

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**Lemma 1 (Degtev).** Let $M$ be a maximal $\Sigma^0_1$ set. Then $UH(M)$.

(Lemma 1 is also mentioned, but not proved, in [3]. In point of fact, Degtev asserts in [1] that $D^*(M)$ holds; his proof, however, stops just short of showing it, the stopping point being $UH(M)$. In the next lemma, we carry the matter one easy step further.)

Recall that an infinite set $C \subseteq N$ is called cohesive if there is no set $W \in \Sigma^0_1$ such that both $C \cap W$ and $C \cap \overline{W}$ are infinite (so that, in particular, maximal $\Sigma^0_1$ sets are just those which have infinite, cohesive complements).

**Lemma 2.** Let $C$ be a cohesive set such that $C \subseteq \overline{M}$ holds for some maximal element $0^\prime$. Then $D^*(C)$.

**Proof.** It is very easily seen that $(\forall S)(\forall T)((D^*(S) \& T \subseteq S) \Rightarrow D^*(T))$; hence, it is enough to show that $D^*(M)$ holds. Now, given $\varphi$, the cohesiveness of $M$ implies that either $M \cap \delta \varphi$ or $M - \delta \varphi$ is finite. If $M \cap \delta \varphi$ is finite, then $[D^*(M)]$ holds "vacuously". If, on the other hand, $M - \delta \varphi$ is finite, then (by reason of the "Reduction Theorem") there is a total recursive function $\varphi_j$ such that $\varphi_j$ and $\varphi_i$ agree on $M \cap \delta \varphi$. By Lemma 1, $[D^*_i(M)]$. Hence, since $\varphi_j$ extends $\varphi_i$ on $M$, $[D^*_i(M)]$. Thus $D^*(M)$, and the lemma is proved.

**Lemma 3 [7].** There exists a maximal $\Sigma^0_1$ set $M$ such that $M < 0^\prime$.

As we noted at the beginning of the proof of Lemma 2, $D^*$ is a hereditary property; this is true also of $D$ and $UH$. The situation with respect to $D^{**}$ is quite different, as our proof of Proposition 6, based on the next lemma, will show. (For background material regarding retraceability, see [2] or [5].)

**Lemma 4.** Let $S$ be an infinite retraceable set such that $\neg D^*(S)$. Then there is a $\Sigma^0_1$ set $C$ such that $S \cap C$ is infinite & $\neg D^{**}(S \cap C)$.

**Proof.** Let $g$ be a partial recursive function which retraces $S$, and let $\varphi = \varphi_o$ be such that $\neg [D^*_0(S)]$. We assume, w.l.o.g., that $g(x) < x$ for all $x \in \delta g$. Let $g^s$, $\varphi^s$ denote, respectively, the sets of pairs belonging to $g$, $\varphi$ after $s$ steps in some fixed recursive enumeration of all pairs in $g$, $\varphi$ (with exactly one pair entering each of $g$, $\varphi$ at each step of the enumeration). We shall enumerate $C$, along with a partial recursive function $\Psi$ having domain $C$, in stages, as follows.

**Stage 0.** Set $C^0 = \Psi^0 = \emptyset$; then proceed to Stage 1.

**Stage s + 1.** For each $x$, let

$$D^*_x = \{ z | x \in \delta \varphi^s - \delta \Psi^s \& x < z \leq \varphi^s(x) \& \langle z, x \rangle \in g^s \}.$$  

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\[ E^s = \{ z \mid (\exists t < s)(\exists x)[\langle z, x \rangle \in g^t \& x \in \delta \Psi^t - \delta \Psi^t \& D_x^t \neq \emptyset \& x < z < \varphi'(x) \& (\forall y < x)[y \notin \delta \Psi^t - \delta \Psi^t \vee D_y^t = \emptyset] \& z \in C^s] \}. \]

If there is no \( x \) such that \( x \in \delta \Psi^t - \delta \Psi^t \& D_x^t \neq \emptyset \), set \( C^{s+1} = C^s \cup E^s \) and \( \Psi^{s+1} = \Psi^s \cup \{ \langle w, 0 \rangle \mid w \in E^s - \delta \Psi^t \} \); then proceed to Stage \( s + 2 \).

Otherwise, let \( x_0 = (\varphi)(x \in \delta \Psi^t - \delta \Psi^t \& D_x^t \neq \emptyset) \), and define:

\[
C_0^{s+1} = C^s \cup \{ x_0 \} \cup D_{x_0}^s; \\
\Psi_0^{s+1} = \Psi^s \cup \{ \langle x_0, \varphi'(x_0) \rangle \}; \\
C^{s+1} = C_0^{s+1} \cup E^s; \\
\Psi^{s+1} = \Psi_0^{s+1} \cup \{ \langle w, 0 \rangle \mid w \in C^{s+1} - \delta \Psi_0^{s+1} \}.
\]

Then proceed to Stage \( s + 2 \).

We define \( C = \bigcup_s C^s \), \( \Psi = \bigcup_s \Psi^s \). Clearly, \( C \in \Sigma_0^1 \) and \( \Psi \) is a recursively enumerable set of pairs such that \( (\forall s)(\exists^1 \Psi^s \subseteq \Psi^{s+1}) \). Since, by a trivial induction on \( s \), each \( \Psi^s \) is seen to be a function, we have that \( \Psi \) is a partial recursive function. Obviously \( \delta \Psi = C \); moreover, if \( x < y \& \{ x, y \} \subseteq C \cap S \& g(y) = x \) \& \( x = p_y(n) \), then \( y = p_x(n + 1) \). The lemma will therefore be proved if we can justify the following claim: there is a sequence \( \{ \langle x_i, y_i \rangle \} \) of pairs such that \( (\forall t)(\exists x_i < y_i < x_{i+1} \& \{ x_i, y_i \} \subseteq S \cap \delta \Psi \& g(y_i) = x_i \& \Psi(x_i) > y_i) \). Suppose we have found the first \( n_0 \) terms, \( \langle x_0, y_0 \rangle, \ldots, \langle x_{n_0-1}, y_{n_0-1} \rangle \), of such a sequence. (If \( n_0 = 0 \), we are starting from scratch.) Let \( s_0 = (\mu)(x \& \{ x_i \mid i < n_0 \} \subseteq \delta \Psi \& g(x) = x \) \& \( x < g(x) \). Let \( \langle x, y \rangle \) be the lexicographically least pair such that: \( \{ x, y \} \subseteq S, \ n_0 > 0 \Rightarrow x > g(x) > y, \ n_0 \), \( \delta \Psi \) \& \( x \notin \delta \Psi \) \& \( y \notin \delta \Psi \). Let \( w_0 = (\mu)(x \in \delta \Psi^t) \). We claim that \( x \in \delta \Psi \). For let \( t_0 = (\mu)(x \in \delta \Psi^t \& \delta \Psi^t \) for some \( s \geq z_0 \). But then, as a trivial induction on \( s \) shows, we have either \( \Psi(x) = 0 \) or \( \Psi(x) = 0 \) if \( \Psi(x) = 0 \), then let \( z_0 = (\mu)(x \in \delta \Psi^t \& \delta \Psi^t \) for some \( s > z_0 \). But then we can define \( x_{n_0} = x, y_{n_0} = y \). If, on the other hand, \( \Psi(x) = 0 \), then, as is clear from the construction, we must have \( g(x) \in \delta \Psi \& \Psi(g(x)) = \Psi(g(x)) \) \& \( g(x) \) \& \( g(x) \). Thus, \( x = \delta \Psi \). By induction, then, the required sequence \( \{ \langle x_i, y_i \rangle \} \) exists and the lemma is proved.

We are now ready to present our "catalog".

**Proposition 1.** \( D^*(S) \Rightarrow D^{**}(S) \Rightarrow UH(S) \).

**Proof.** Obvious, from definitions.

**Proposition 2.** If \( S \) is regressive, then \( D^*(S) \Rightarrow D(S) \).
Proof. It is clear that any regressive set satisfying condition $D^*$ is in fact retraceable. Now use [5, proof of Theorem 3.2].

Proposition 2 might seem a bit strange at first sight, since $[D_i(S)]$ involves the action of $\varphi_i$ on $S$ while $[D^*_i(S)]$ does not. The next proposition redresses the intuitive balance.

**Proposition 3.** There exists a $\Pi^0_1$ set $S$ such that $D^*(S) \land \neg D(S)$.

**Proof.** As shown in [8], $D(S) \Rightarrow S > \emptyset'$. Applying Lemma 3, let $S$ be a cohesive $\Pi^0_1$ set such that $S < \emptyset'$. Then $\neg D(S)$. On the other hand, $D^*(S)$ holds by Lemma 2.

**Proposition 4.** There exists a retraceable $\Pi^0_1$ set $S$ such that $D(S) \land \neg UH(S)$.

**Proof.** By (for instance) [5, Theorems 3.1 and 3.2], there is a retraceable $\Pi^0_1$ set $T$ such that $D(T)$ holds. By [6, Theorem 4.1], there is a second retraceable $\Pi^0_1$ set $R$ such that $p_T \circ p_R$ is the principal function of a set $S$ for which $\neg UH(S)$. But, the condition $D$ is (as is very easily seen) preserved under compositional injection; and, the composition of principal functions of two infinite retraceable $\Pi^0_1$ sets is again an infinite retraceable $\Pi^0_1$ set. $S$ therefore verifies our proposition. (Easy direct constructions also are available for proving Proposition 4.)

**Proposition 5.** There exists a retraceable $\Pi^0_1$ set $S$ such that $D^{**}(S) \land \neg D^*(S)$.

**Proof.** By [2], [4], and [7], let $S$ be a retraceable $\Pi^0_1$ set such that $S < \emptyset' \land D^{**}(S)$. By [8] plus Proposition 2, we have $\neg D^*(S)$.

**Proposition 6.** There exists an infinite d.r.e. set $S$ such that $UH(S) \land \neg D^{**}(S)$.

**Proof.** Applying Proposition 5, let $S_0$ be a retraceable $\Pi^0_1$ set such that $\neg D^*(S_0) \land D^{**}(S_0)$. Applying Lemma 4, let $C$ be a $\Sigma^0_1$ set such that $S_0 \cap C$ is infinite $\land \neg D^{**}(S_0 \cap C)$. Since $UH$ is a hereditary condition, and since $UH$ and $D^{**}$ are equivalent for $\Pi^0_1$ sets (using the “Reduction Theorem”), we see that $S = S_0 \cap C$ verifies the proposition.

Several fairly obvious questions occur in connection with the foregoing results:

**Q1.** Is there a cohesive set $C$ such that $\neg D^*(C)$?

**QII.** Is there a complete maximal $\Sigma^0_1$ set $M$ such that $\neg D(\overline{M})$?

**QIII.** If $S$ is $\Pi^0_1$, can the set $S \cap C$ of Lemma 4 (and hence the set $S$ of Proposition 6) be required to be retraceable (or even, merely, regressive)? In an earlier version of this paper, we claimed this could be done. The referee, however, spotted a formidable gap in the proof; retraceability was lost during repairs.

**QIV.** Is Lemma 4 a nonvacuous assertion? That is, is there an example of...
an infinite set $S$ such that $\neg D^*(S) \& (\forall C \in \Sigma_1)[S \cap C \text{ infinite } \Rightarrow D^{**}(S \cap C)]$?

REFERENCES


DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409