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A NOTE ON KELLOGG'S UNIQUENESS THEOREM FOR FIXED POINTS

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ABSTRACT. In 1975, R. B. Kellogg gave a condition guaranteeing uniqueness for the fixed point whose existence is insured by the Schauder Theorem. In this note, we indicate how to extend Kellogg's result to the class of k-set-contractions.

R. B. Kellogg [**1**] established the following uniqueness criterion for the Schauder Fixed Point Theorem:

THEOREM. *Let D be a bounded convex open subset of a real Banach space X, and let F: $\overline{D} \to \overline{D}$ be a compact continuous map which is continuously Fréchet differentiable on D. Suppose that* (a) *for each $x \in D$, 1 is not an eigenvalue of $F'(x)$, and* (b) *for each $x \in \partial D$, $x \neq F(x)$. Then F has a unique fixed point.*

In this note we indicate how to extend Kellogg's theorem to a larger class of maps than the compact ones.

In order to describe the class of maps with which we work, let X be a real Banach space. If $B \subseteq X$ is bounded, we define the set measure of noncompactness of B, $\alpha(B)$, by

$$\alpha(B) = \inf\{\varepsilon > 0\text{: } B \text{ has a finite cover by sets whose diameters do not exceed } \varepsilon\}.$$

Clearly, B is precompact iff $\alpha(B) = 0$.

A function F whose domain is a subset of X is called a k-set-contraction, or an α-Lipschitz operator, if there is a nonnegative constant k such that $\alpha(F(B)) \leqslant k\alpha(B)$ for every bounded subset B of the domain of F. It is known that the Schauder Fixed Point Theorem extends to the class of k-set-contractions for which $k < 1$. See, e.g., [**3**, Theorem IV. 3.2, p. 125].

It is also known that a Fréchet derivative of a k-set-contraction is a k_1-set-contraction for some $k_1 \leqslant k$. See [**3**, Proposition III.6.5, p. 89].

In 1969, R. D. Nussbaum established that if F is a linear k-set-contraction, $k < 1$, then $\{\lambda \in \mathbf{C}\text{: } |\lambda| \geqslant 1 \text{ and } \lambda \in \sigma(F)\}$ is a finite set each of whose elements is an eigenvalue of F of finite multiplicity. See [**4**].

In 1970, the same author extended the degree theory for Banach spaces to a

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class of operators which includes the k-set-contractions for which $k < 1$. See **[5]**. At about the same time, B. N. Sadovskii independently developed a degree theory which includes the k-set-contractions in Banach spaces for which $k < 1$. See **[6]**.

In 1971, C. A. Stuart and J. F. Toland extended Theorem II.4.7, p. 136 of **[2]** to the k-set-contractions for which $k < 1$. See **[7]**.

We now note that in the proof of his Lemma, Kellogg uses the compactness hypothesis only to determine the structures of the spectra of the compact operators involved. By appealing to Nussbaum's result, cited above, concerning the spectrum of a linear k-set-contraction, we can use Kellogg's argument to extend his Lemma to linear k-set-contractions.

We are now ready for our principal result.

THEOREM. *Let D be a bounded convex open subset of a real Banach space X, and let F: $\bar{D} \to \bar{D}$ be a continuous k-set-contraction for which $k < 1$. Assume that F is continuously Fréchet differentiable on D. Suppose that* (a) *for each $x \in D$, 1 is not an eigenvalue of $F'(x)$, and* (b) *for each $x \in \partial D$, $x \neq F(x)$. Then F has a unique fixed point.*

PROOF. We need only make some minor modifications of Kellogg's argument. First, we establish that the fixed point set K of the k-set-contraction $(k < 1)$ F must have a limit point in D if it is not a finite set. This is immediate from the k-set-contractive hypothesis because K is a closed bounded set and $\alpha(K) = \alpha(F(K)) \leqslant k\alpha(K)$, whence K is compact in D (recall that $k < 1$ and $K \cap \partial D = \varnothing$).

For the rest of the proof we simply follow Kellogg's argument, using the results cited above and concerning k-set-contractions in place of the analogous results cited by Kellogg and concerning compact maps.

REMARK. As was pointed out to the author by the referee, the crucial point of Kellogg's argument is the establishment of the fact that $\deg(0, I - F, D) = 1$. This fact can be established under a variety of conditions which are more general than the condition that the open set D be convex bounded, and $F(\bar{D}) \subseteq \bar{D}$. For example, we can replace this assumption with one of the following assumptions:

(a) D is convex, bounded, and $F(\partial D) \subseteq \bar{D}$. (See **[5**, Corollary 5 to Proposition 1, §E, p. 247].)

(b) $\bar{D} \cap F(\bar{D})$ is bounded, and there is a point $x_0 \in D$ such that either $\|F(x) - x_0\| < \|x - x_0\|$ for all $x \in \partial D$ or $\|x_0 - F(x)\| < \|x - F(x)\|$ for all $x \in \partial D$. (See **[6**, Theorem 3.3.4, p. 138]–which is Rouché's Theorem!)

(c) $\bar{D} \cap F(\bar{D})$ is bounded, and there is a point $x_0 \in D$ such that $F(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda \geqslant 1$ and $x \in \partial D$. (See **[6**, Theorem 3.3.5, p. 139].)

It should be further noted that either of conditions (b) or (c) renders the assumption that F has no fixed points in ∂D redundant.

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