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THE ISOPERIMETRIC THEOREM FOR CURVES ON MINIMAL SURFACES

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ABSTRACT. A short proof is given for a sharpened form of the isoperimetric inequality for curves on minimal surfaces.

By following a line of development used by Sachs [7] in treating inequalities for plane curves, one can give an economical formulation to the proof of the isoperimetric theorem for curves on minimal surfaces.

Let C be a smooth simple closed curve in Euclidean n-space, where $n \geqslant 2$. In what follows we shall assume, as can be achieved by a translation, that the centroid of arc length of C is at the origin. Hence, if x is the position vector, we assume

$$\int_C x\, ds = 0. \tag{1}$$

If we let $y(t) = x(Lt/2\pi)$, $0 \leqslant t \leqslant 2\pi$, where L is the length of C, then (1) allows us to apply Wirtinger's inequality [**1**, p. 105] componentwise to obtain

$$\int_0^{2\pi} |y|^2\, dt \leqslant \int_0^{2\pi} \left| \frac{dy}{dt} \right|^2 dt, \tag{2}$$

from which follows

$$\int_C |x|^2\, ds \leqslant \frac{L^2}{4\pi^2} \int_C \left| \frac{dx}{ds} \right|^2 ds = \frac{L^3}{4\pi^2}. \tag{3}$$

Denoting the integral on the far left-hand side of (3) by I, we have then an inequality of Sachs [7],

$$L^3 - 4\pi^2 I \geqslant 0. \tag{4}$$

Now, with the same hypotheses as above, suppose C is the boundary of a smooth orientable surface S with area A. Then we have the area formula (see [3], [4], [5])

$$\int_C x \cdot \nu\, ds = 2A + 2\iint_S x \cdot H\, dA, \tag{5}$$

where ν is the outward pointing unit normal to C tangent to S, and H is the mean curvature vector of S, which satisfies $H \equiv 0$ in case S is a minimal

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surface. In analogy to the procedure of Sachs [7] we form the vector field $D = x - (L/2\pi)\nu$ along C and define the functional

$$R^2 = \frac{1}{L} \int_C D \cdot D \, ds = \frac{I}{L} - \frac{1}{\pi} \int_C x \cdot \nu \, ds + \frac{L^2}{4\pi^2}. \tag{6}$$

Substitution from (5) into (6) and rearrangement of terms gives

$$L^2 - 4\pi A = 2\pi^2 R^2 + \frac{1}{2L}(L^3 - 4\pi^2 I) + 4\pi \iint_S x \cdot H \, dA. \tag{7}$$

Applying (4), we have

$$L^2 - 4\pi A \geqslant 2\pi^2 R^2 + 4\pi \iint_S x \cdot H \, dA, \tag{8}$$

giving, when $H \equiv 0$, a sharpened form of the isoperimetric inequality for curves on minimal surfaces (see [2]–[6]).

When S is a minimal surface, (7) yields

$$4\pi^2 I - 8\pi LA + L^3 = 4\pi^2 LR^2 \geqslant 0, \tag{9}$$

which can be rearranged as

$$L^2 - 4\pi A \geqslant (4\pi/L)(LA - \pi I). \tag{10}$$

On the other hand, (4) can be written in the form

$$L^2 - 4\pi A \geqslant (4\pi/L)(\pi I - LA). \tag{11}$$

From (10) and (11) we have another sharpening of the isoperimetric inequality for curves on minimal surfaces,

$$L^2 - 4\pi A \geqslant (4\pi/L)|\pi I - LA|, \tag{12}$$

an inequality established in the same fashion for plane curves by Sachs [7].

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