ON THE LOCALIZATION OF THE SPECTRUM FOR SYSTEMS OF OPERATORS

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Abstract. Let \( a = (a_1, \ldots, a_n) \) be a commuting system of linear continuous operators on a complex Banach space \( X \). We show that, for any \( x \in X \), the local analytic spectrum \( \sigma(a, x) \) [1] is contained in the spectral hull of the local spectrum \( \text{sp}(a, x) \) [4].

Introduction. In this paper we shall define more accurately the relation between two notions of local spectrum for systems of operators.

Let \( X \) be a complex Banach space and let \( a = (a_1, \ldots, a_n) \) be a commuting system \((n\text{-tuple})\) of linear continuous operators on \( X \).

The simplest way to define the spectrum of an arbitrary element \( x \in X \), with respect to \( a \), is the following. Consider the union \( \rho(a, x) \) of all open sets \( D \subset \mathbb{C}^n \) such that there exist \( n \) analytic functions \( f_1, \ldots, f_n : D \to X \), satisfying \( \sum_{i=1}^{n} (z_i - a_i) f_i(z) = x, \ z \in D \). Then the spectrum of \( x \) with respect to \( a \) is the set \( \sigma(a, x) = \mathbb{C}^n \setminus \rho(a, x) \). This notion of local spectrum was used in [1] for the study of spectral decompositions dependent on functional calculi.

In [4] the author has studied spectral decompositions not necessarily dependent on functional calculi and has found very useful to define another notion of local spectrum. Generally speaking, the spectrum of \( x \) with respect to \( a \) in our sense, denoted by \( \text{sp}(a, x) \), is the support of a certain differential form (or rather of a cohomology class of such forms). This notion was suggested by a new definition of the functional analytic calculus for several commuting operators, proposed in [7].

We have proved in [4] that \( \text{sp}(a, x) \subset \sigma(a, x) \) for any \( x \in X \).

The main result of the present paper is that \( \sigma(a, x) \) is contained in the spectral hull of \( \text{sp}(a, x) \), for any \( x \in X \).

1. Preliminaries. We shall recall the definition of the Cauchy-Weil integral which will be our main tool in what follows (for details, see [7, §3] and [4, Preliminaries]).

Let \( X \) be a complex Banach space and \( a = (a_1, \ldots, a_n) \) be a commuting system of linear continuous operators on \( X \). Denote by \( \text{sp}(a, X) \) the Taylor spectrum of \( a \) on \( X \) [6].
Let $U$ be an open neighbourhood of $\text{sp}(a, X)$ and $f$ be an $X$-valued analytic function on $U$. Denote by $\mathcal{B}(U, X)$ the space of all continuous $X$-valued functions on $U$ which are infinitely differentiable, in the sense of distributions, with respect to $\bar{z}_1, \ldots, \bar{z}_n$ [7, §2]. Consider a system $\sigma = (s_1, \ldots, s_n)$ of indeterminates and denote by $\Lambda^p[\sigma \cup \partial \bar{z}, \mathcal{B}(U, X)]$ the space of all forms of degree $p$ in the indeterminates $s_1, \ldots, s_n, \partial \bar{z}_1, \ldots, \partial \bar{z}_n$ having the coefficients in $\mathcal{B}(U, X)$. Denote by $\alpha \oplus \bar{\partial}$ the (coboundary) operator defined by

\[
(\alpha \oplus \bar{\partial})\psi(z) = \left[ (z_1 - a_1)s_1 + \cdots + (z_n - a_n)s_n
+ (\partial / \partial \bar{z}_1) \partial \bar{z}_1 + \cdots + (\partial / \partial \bar{z}_n) \partial \bar{z}_n \right] \wedge \psi(z), \quad z \in U,
\]

where $\psi \in \Lambda^p[\sigma \cup \partial \bar{z}, \mathcal{B}(U, X)]$.

The Cauchy-Weil integral of $f$ with respect to $a$ (on $U$) is an element of $X$ obtained as follows. Since $U$ is an open neighbourhood of $\text{sp}(a, X)$, the cohomology class of the form of $sf = fs_1 \wedge \cdots \wedge s_n$, with respect to $\alpha \oplus \bar{\partial}$, contains a form $\chi$ with compact support (i.e. there exist a form $\chi$ with compact support and another form $\psi$ such that $sf = \chi - (\alpha \oplus \bar{\partial})\psi$). Denote by $\pi \chi$ that part of $\chi$ containing only the indeterminates $\partial \bar{z}_1, \ldots, \partial \bar{z}_n$. That will be of the form $\pi \chi = h \partial \bar{z}_1 \wedge \cdots \wedge \partial \bar{z}_n$, where $h \in \mathcal{B}(U, X)$ and $h$ has compact support.

The Cauchy-Weil integral of $f$ with respect to $a$ (on $U$) is defined by

\[
\int_U R_{z-a}f(z) \wedge dz_1 \wedge \cdots \wedge dz_n \quad (1)
\]

\[
= \int_U (-1)^n \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n.
\]

The value of the integral on the right side depends only on the cohomology class of $\chi$ (as form with compact support) with respect to $\alpha \oplus \bar{\partial}$. Moreover, it does not change if $U$ becomes smaller and depends continuously on the function $f$.

With the Cauchy-Weil integral at hand, it is easy to define the functional analytic calculus. If $f$ is a scalar function analytic in an open neighbourhood of $\text{sp}(a, X)$, then [7, §4] $f(a)$ is defined by

\[
f(a)x = \frac{1}{(2\pi i)^n} \int_U R_{z-a}f(z)x \wedge dz_1 \wedge \cdots \wedge dz_n, \quad x \in X.
\]

In particular, taking into account that $1(a) = \text{id}$, we obtain

\[
x = \frac{1}{(2\pi i)^n} \int_U R_{z-a}x \wedge dz_1 \wedge \cdots \wedge dz_n, \quad x \in X.
\]

2. The relation between $\sigma(a, x)$ and $\text{sp}(a, x)$. We may now define the local spectrum $\text{sp}(a, x)$ as the support of the integrand from (3), having in view to obtain a local variant of this formula.

**Definition 1** [4]. The resolvent set $r(a, x)$ of $x$ with respect to $a$ is the union...
of all open sets $D$ with the property that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathcal{B}(D, X)]$, satisfying $sx = (\alpha \oplus \bar{\delta})\psi$. The spectrum $\sigma(a, x)$ of $x$ with respect to $a$ is the complement in $\mathbb{C}^n$ of $r(a, x)$, $\sigma(a, x) = \mathbb{C}^n \setminus r(a, x)$.

If $n = 1$ then the form $\psi$ in Definition 1 is of degree zero and, consequently, it is simply a function from $\mathcal{B}(D, X)$. Moreover, the relation $sx = (\alpha \oplus \bar{\delta})\psi$ is equivalent to $x = (z - a)\psi(z)$, $z \in D$ and $(\partial / \partial \bar{z})\psi = 0$. Therefore $\psi$ is an analytic function on $D$ and, consequently, we have, in this case, $\sigma(a, x) = \sigma(a, x)$.

If the equation in $\psi$, $sx = (\alpha \oplus \bar{\delta})\psi$, has a global solution defined on $r(a, x)$ then, for any open neighbourhood $V$ of $\sigma(a, x)$, the cohomology class of $sx$ with respect to $\alpha \oplus \bar{\delta}$ contains a form $\chi$ with compact support in $V$ [4, §1] and, consequently, we get the following local variant of (3):

$$x = \frac{1}{(2\pi i)^n} \int_V (-1)^n \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n.$$  

Just this local variant of (3) motivates our Definition 1. It is easy to see that, in the case $n = 1$, the equation $sx = (\alpha \oplus \bar{\delta})\psi$ has a global solution if and only if the operator $a$ has the single-valued extension property [3], [2].

DEFINITION 2. The system $a$ is said to have the localization property (in short, property L) if $H^{n-1}(\mathcal{B}(G, X), \alpha \oplus \bar{\delta}) = 0$ for any open set $G \subset \mathbb{C}^n$.

We have denoted by $H^{n-1}(\mathcal{B}(G, X), \alpha \oplus \bar{\delta})$ the cohomology module of order $n - 1$ for the cochain complex consisting of the spaces $\Lambda^p[\sigma \cup d\bar{z}, \mathcal{B}(G, X)]$ and of $\alpha \oplus \bar{\delta}$ as coboundary operator.

Let $U$ be an open set in $\mathbb{C}^n$ and let us denote by $\mathcal{U}(U, X)$ the space of all $X$-valued analytic functions on $U$. Denote by $a$ the operator defined by

$$(\alpha \psi)(z) = \left[ (z_1 - a_1)s_1 + \cdots + (z_n - a_n)s_n \right] \wedge \psi(z), \quad z \in U,$$

where $\psi$ is an exterior form in $s$ with coefficients in $\mathcal{U}(U, X)$. Then the object consisting of the spaces $\Lambda^p[\sigma, \mathcal{U}(U, X)]$ and of $\alpha$ as coboundary operator is a cochain complex. We shall denote by $H^i(\mathcal{U}(U, X), \alpha)$ the cohomology modules of this complex.

DEFINITION 3. The system $a$ is said to have the analytic localization property (in short property LA) if $H^i(\mathcal{U}(U, X), \alpha) = 0$, $0 \leq i \leq n - 1$, for any open polydisc $U \subset \mathbb{C}^n$.

We have proved in [4] that property LA implies property L. If $n = 1$ then the properties are equivalent to the single-valued extension property.

DEFINITION 4 [7]. Let $K \subset \mathbb{C}^n$ be a compact set. The spectral hull of $K$ is the set of all elements $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ with the property that the equation

$$(z_1 - w_1)f_1(z) + \cdots + (z_n - w_n)f_n(z) = 1$$

has no analytic solutions $f_1, \ldots, f_n$ in any open neighbourhood of $K$.

We shall prove that, for any $x \in X$, $\sigma(a, x)$ is contained in the spectral hull of $\sigma(a, x)$. For that we need two lemmas.

LEMMA 1. Let $K$ be a compact subset of $\mathbb{C}^n$. If $w = (w_1, \ldots, w_n)$ does not belong to the spectral hull of $K$ then there exist an open neighbourhood $U$ of $K$,
an open neighbourhood $V$ of $w$ and $n$ scalar functions $f_1(z, \xi), \ldots, f_n(z, \xi)$ analytic on $U \times V$, satisfying

$$\sum_{j=1}^{n} (z_j - \xi_j)f_j(z, \xi) \equiv 1, \quad (z, \xi) \in U \times V.$$ 

**Proof.** Since $w$ does not belong to the spectral hull of $K$, it follows, according to Definition 4, that there exist an open neighbourhood $U_1$ of $K$ and $n$ scalar analytic functions $f_1, \ldots, f_n$ on $U_1$, such that

$$\sum_{j=1}^{n} (z_j - w_j)f_j(z) \equiv 1, \quad z \in U_1.$$ 

From here we obtain

$$\sum_{j=1}^{n} (z_j - \xi_j)f_j(z) = \sum_{j=1}^{n} (z_j - w_j)f_j(z) + \sum_{j=1}^{n} (w_j - \xi_j)f_j(z)$$

$$= 1 + \sum_{j=1}^{n} (w_j - \xi_j)f_j(z), \quad z \in U_1.$$ 

Let $U$ be an open relatively compact neighbourhood of $K$ such that $U \subseteq U_1$. It is easy to see that there exists an open neighbourhood $V$ of $w$ such that $|\sum_{j=1}^{n} (w_j - \xi_j)f_j(z)| < 1$ for any $\xi \in V$ and any $z \in U$. Thus, for any $(z, \xi) \in U \times V$, the complex number

$$\sum_{j=1}^{n} (z_j - \xi_j)f_j(z) = 1 + \sum_{j=1}^{n} (w_j - \xi_j)f_j(z)$$

is different from zero. Denoting

$$h(z, \xi) = \left[ \sum_{j=1}^{n} (z_j - \xi_j)f_j(z) \right]^{-1}, \quad (z, \xi) \in U \times V,$$

we shall obtain an analytic function on $U \times V$ satisfying

$$h(z, \xi)\left[ \sum_{j=1}^{n} (z_j - \xi_j)f_j(z) \right] \equiv 1, \quad (z, \xi) \in U \times V.$$ 

Therefore we may define

$$f_j(z, \xi) = h(z, \xi)f_j(z), \quad (z, \xi) \in U \times V, 1 \leq j \leq n,$$

and the statement of Lemma 1 follows.

**Lemma 2.** Suppose that the system $a$ satisfies $L$. Let $U$ be an open neighbourhood of the set $\text{sp}(a, x)$ and $f$ a scalar analytic function on $U$. Then, for any form appearing in (3) and any number $j, 1 \leq j \leq n$, we have

$$\int_{U} (-1)^{n} f(z)(z_j - a_j)\pi \wedge dz_1 \wedge \cdots \wedge dz_n = 0.$$ 

**Proof.** We shall apply the theorem of Stokes. Let $j$ be a fixed number, $1 \leq j \leq n$, and let us denote by $\tilde{a}_j$ the operator defined by $(\tilde{a}_j\psi)(z) = (z_j - a_j)\psi(z), z \in U$, where $\psi$ is a form in $s$ and $d\xi$ having coefficients in $\mathcal{B}(U, X)$. 

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Let $\chi$ be an arbitrary form from those appearing in $(3')$. Then there exists a form $\psi^*$ on $U$ such that $s\chi - \chi = (\alpha \Theta \delta)\psi^*$. By applying the operator $\bar{a}_j$ to both sides of this equality and multiplying by $f$, we obtain
\[ s\bar{a}_j\chi - f\bar{a}_j\chi = (\alpha \oplus \bar{\delta})(f\bar{a}_j\psi^*). \]

On the other hand, denoting
\[ \psi_j(z) = f(z)x_{s_1} \wedge \cdots \wedge \delta_j \wedge \cdots \wedge s_n, \quad z \in U, \]
we have $s\bar{a}_j\chi = (\alpha \oplus \bar{\delta})\psi_j$, whence
\[ f\bar{a}_j\chi = (\alpha \oplus \bar{\delta})(\psi_j - f\bar{a}_j\psi^*). \]

Taking into account that $\chi$ has compact support in $U$, it follows that $f\bar{a}_j\chi$ has compact support in $U$. By using property L we shall prove that $\psi_j = (\alpha \oplus \bar{\delta})\psi_j$ on $U$, where $\psi_j$ has also compact support. With this result at hand, we get
\[ f\bar{a}_j\pi\chi = \pi(f\bar{a}_j\chi) = \pi(\alpha \oplus \bar{\delta})\psi_j = \tilde{\psi}_j(\pi\psi_j). \]

From here we obtain
\[
\begin{align*}
\int_U (-1)^n f(z)(z_j - a_j)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
= \int_U (-1)^n f\bar{a}_j\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
= \int_U (-1)^n \tilde{\psi}_j(\pi\psi_j) \wedge dz_1 \wedge \cdots \wedge dz_n \\
= \int_U (-1)^n d(\pi\psi_j \wedge dz_1 \wedge \cdots \wedge dz_n).
\end{align*}
\]

Taking into account that $\psi_j$ has compact support in $U$ and applying the theorem of Stokes, we deduce that the last integral is equal to zero and, consequently,
\[ \int_U (-1)^n f(z)(z_j - a_j)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n = 0. \]

Therefore it remains to prove that there exists a form $\psi_j$ with compact support such that $f\bar{a}_j\chi = (\alpha \oplus \bar{\delta})\psi_j$. Denote by $K$ the support of $\chi$ (which is a compact set). Since $f\bar{a}_j\chi = (\alpha \oplus \bar{\delta})(\psi_j - f\bar{a}_j\psi^*)$ and $\chi = 0$ on $U \setminus K$, we have $(\alpha \oplus \bar{\delta})(\psi_j - f\bar{a}_j\psi^*) = 0$ on $U \setminus K$. Thus, by applying property L for $G = U \setminus K$, there exists a form $\varphi_j$ such that $\psi_j - f\bar{a}_j\psi^* = (\alpha \oplus \bar{\delta})\varphi_j$. Consider now two open relatively compact sets $U_1$ and $U_2$ such that $K \subset U_1 \subset \bar{U}_1 \subset U_2 \subset \bar{U}_2 \subset U$. Let $h$ be a $C^\infty$ scalar function on $U$ such that $h = 0$ on $U_1$ and $h = 1$ on $U \setminus U_2$. Let us define the form $\tilde{\psi}_j$ by
\[ \tilde{\psi}_j = h\psi_j \quad \text{on} \quad U \setminus K, \]
\[ = 0 \quad \text{on} \quad U_1. \]

Denoting $\psi_j = (\psi_j - f\bar{a}_j\psi^*) - (\alpha \oplus \bar{\delta})\tilde{\psi}_j$, we obtain
On the other hand, we have $\tilde{\psi}_j = h\psi_j$ on $U \setminus K$ and $h = 1$ on $U \setminus U_2$, whence $\tilde{\psi}_j = 0$ on $U \setminus U_2$. This completes the proof.

We can prove now our main result concerning the relation between the spectrum $\sigma(a, x)$ defined in the Introduction and the spectrum $\text{sp}(a, x)$.

**Theorem.** Let $a$ be a commuting system of operators satisfying property L. Then, for any $x \in X$, the analytic spectrum $\sigma(a, x)$ is contained in the spectral hull of the spectrum $\text{sp}(a, x)$.

**Proof.** We shall prove that if $w$ does not belong to the spectral hull of $\text{sp}(a, x)$ then $w \in \rho(a, x)$, from where the Theorem will obviously follow. According to Lemma 1, there exist an open neighbourhood $U$ of $\text{sp}(a, x)$, an open neighbourhood $V$ of $w$ and $n$ scalar functions $f_1(z, \xi), \ldots, f_n(z, \xi)$, analytic on $U \times V$ and satisfying

$$
\sum_{j=1}^{n} (z_j - \xi_j)f_j(z, \xi) = 1, \quad (z, \xi) \in U \times V.
$$

Let us define the $X$-valued functions $f_j$ on $V$ by

$$
f_j(\xi) = -\frac{1}{(2\pi i)^n} \int_U (-1)^n f_j(z, \xi) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n, \quad 1 \leq j \leq n,
$$

where $\chi$ is one of the forms appearing in $(3')$. Since the form $\chi$ has compact support, the forms appearing under the integral also have compact support. Moreover, since $f_j(z, \xi)$ are analytic functions, $1 \leq j \leq n$, it is easy to see that $f_j$ are analytic functions on $V$, $1 \leq j \leq n$. We shall prove that $\sum_{j=1}^{n} (\xi_j - a_j)f_j(\xi) = x$, $\xi \in V$, and thus it will follow that $w \in \rho(a, x)$, as desired.

By applying Lemma 2 for the functions $f_j(z, \xi)$, we get

$$
\int_U (-1)^n f_j(z, \xi)(z_j - a_j) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n = 0, \quad 1 \leq j \leq n.
$$

Consequently, we have

$$
\sum_{j=1}^{n} (\xi_j - a_j)f_j(\xi) = -\frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (\xi_j - a_j)f_j(z, \xi) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n
$$

$$
= -\frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (\xi_j - z_j)f_j(z, \xi) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n
$$

$$
- \frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (z_j - a_j)f_j(z, \xi) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n
$$

$$
= \frac{1}{(2\pi i)^n} \int_U (-1)^n \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n = x.
$$

For the last equality we have used $(3')$. Therefore
and the proof is finished.

**Remark.** It is possible to define a local spectrum similar to \( \text{sp}(a, x) \) by using the space \( C^\infty \) instead of the space \( \mathcal{B} \). It is clear that \( \text{sp}(a, x) \) will be contained in this spectrum. All the results proved in [4] for \( \text{sp}(a, x) \) also remain valid for this new spectrum. Indeed, the properties of the space \( \mathcal{B} \) used in [4] are still valid for the space \( C^\infty \) (see also [5]).

**References**


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