PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 2, May 1978

ON THE LOCALIZATION OF THE SPECTRUM FOR SYSTEMS OF OPERATORS

ŞTEFAN FRUNZĂ

ABSTRACT. Let $a = (a_1, \ldots, a_n)$ be a commuting system of linear continuous operators on a complex Banach space X. We show that, for any $x \in X$, the local analytic spectrum $\sigma(a, x)$ [1] is contained in the spectral hull of the local spectrum $\mathrm{sp}(a, x)$ [4].

Introduction. In this paper we shall define more accurately the relation between two notions of local spectrum for systems of operators.

Let X be a complex Banach space and let $a = (a_1, \ldots, a_n)$ be a commuting system (n-tuple) of linear continuous operators on X.

The simplest way to define the spectrum of an arbitrary element $x \in X$, with respect to a, is the following. Consider the union $\rho(a, x)$ of all open sets $D \subset C^n$ such that there exist n analytic functions $f_1, \ldots, f_n: D \to X$, satisfying $\sum_{i=1}^n (z_i - a_i) f_i(z) \equiv x$, $z \in D$. Then the spectrum of x with respect to a is the set $\sigma(a, x) = C^n \setminus \rho(a, x)$. This notion of local spectrum was used in [1] for the study of spectral decompositions dependent on functional calculi.

In [4] the author has studied spectral decompositions not necessarily dependent on functional calculi and has found very useful to define another notion of local spectrum. Generally speaking, the spectrum of x with respect to a in our sense, denoted by $\mathrm{sp}(a, x)$, is the support of a certain differential form (or rather of a cohomology class of such forms). This notion was suggested by a new definition of the functional analytic calculus for several commuting operators, proposed in [7].

We have proved in [4] that $\mathrm{sp}(a, x) \subset \sigma(a, x)$ for any $x \in X$.

The main result of the present paper is that $\sigma(a, x)$ is contained in the spectral hull of $\mathrm{sp}(a, x)$, for any $x \in X$.

1. Preliminaries. We shall recall the definition of the Cauchy-Weil integral which will be our main tool in what follows (for details, see [7, §3] and [4, Preliminaries]).

Let X be a complex Banach space and $a = (a_1, \ldots, a_n)$ be a commuting system of linear continuous operators on X. Denote by $\mathrm{sp}(a, X)$ the Taylor spectrum of a on X [6].

Received by the editors May 17, 1977.

AMS (MOS) subject classifications (1970). Primary 47D99; Secondary 32C30, 32C35.

Key words and phrases. Commuting system of operators, spectrum, local spectrum, differential form, cohomology, spectral hull, Stokes' theorem.

Let U be an open neighbourhood of $\mathrm{sp}(a, X)$ and f be an X-valued analytic function on U. Denote by $\mathfrak{B}(U, X)$ the space of all continuous X-valued functions on U which are infinitely differentiable, in the sense of distributions, with respect to $\bar{z}_1, \ldots, \bar{z}_n$ [7, §2]. Consider a system $\sigma = (s_1, \ldots, s_n)$ of indeterminates and denote by $\Lambda^p[\sigma \cup d\bar{z}, \mathfrak{B}(U, X)]$ the space of all forms of degree p in the indeterminates $s_1, \ldots, s_n, d\bar{z}_1, \ldots, d\bar{z}_n$ having the coefficients in $\mathfrak{B}(U, X)$. Denote by $\alpha \oplus \bar{\partial}$ the (coboundary) operator defined by

$$\left[(\alpha \oplus \bar{\partial})\psi\right](z) = \left[(z_1 - a_1)s_1 + \cdots + (z_n - a_n)s_n + (\partial/\partial\bar{z}_1)\, d\bar{z}_1 + \cdots + (\partial/\partial\bar{z}_n)\, d\bar{z}_n\right] \wedge \psi(z), \qquad z \in U,$$

where $\psi \in \Lambda^p[\sigma \cup d\bar{z}, \mathfrak{B}(U, X)]$.

The Cauchy-Weil integral of f with respect to a (on U) is an element of X obtained as follows. Since U is an open neighbourhood of $\mathrm{sp}(a, X)$, the cohomology class of the form of $sf = f s_1 \wedge \cdots \wedge s_n$, with respect to $\alpha \oplus \bar{\partial}$, contains a form χ with compact support (i.e. there exist a form χ with compact support and another form ψ such that $sf - \chi = (\alpha \oplus \bar{\partial})\psi$). Denote by $\pi\chi$ that part of χ containing only the indeterminates $d\bar{z}_1, \ldots, d\bar{z}_n$. That will be of the form $\pi\chi = h\, d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$, where $h \in \mathfrak{B}(U, X)$ and h has compact support.

The Cauchy-Weil integral of f with respect to a (on U) is defined by

$$\int_U R_{z-a} f(z) \wedge dz_1 \wedge \cdots \wedge dz_n = \int_U (-1)^n \pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n. \tag{1}$$

The value of the integral on the right side depends only on the cohomology class of χ (as form with compact support) with respect to $\alpha \oplus \bar{\partial}$. Moreover, it does not change if U becomes smaller and depends continuously on the function f.

With the Cauchy-Weil integral at hand, it is easy to define the functional analytic calculus. If f is a scalar function analytic in an open neighbourhood of $\mathrm{sp}(a, X)$, then [7, §4] $f(a)$ is defined by

$$f(a)x = \frac{1}{(2\pi i)^n} \int_U R_{z-a} f(z) x \wedge dz_1 \wedge \cdots \wedge dz_n, \qquad x \in X. \tag{2}$$

In particular, taking into account that $1(a) = \mathrm{id}$, we obtain

$$x = \frac{1}{(2\pi i)^n} \int_U R_{z-a} x \wedge dz_1 \wedge \cdots \wedge dz_n, \qquad x \in X. \tag{3}$$

2. The relation between $\sigma(a, x)$ and $\mathrm{sp}(a, x)$. We may now define the local spectrum $\mathrm{sp}(a, x)$ as the support of the integrand from (3), having in view to obtain a local variant of this formula.

DEFINITION 1 [4]. *The resolvent set $r(a, x)$ of x with respect to a is the union*

of all open sets D with the property that there exists a form $\psi \in \Lambda^{n-1}[\sigma \cup d\bar{z}, \mathfrak{B}(D, X)]$, satisfying $sx = (\alpha \oplus \bar{\partial})\psi$. The spectrum $\mathrm{sp}(a, x)$ *of x with respect to a is the complement in C^n of $r(a, x)$,* $\mathrm{sp}(a, x) = C^n \setminus r(a, x)$.

If $n = 1$ then the form ψ in Definition 1 is of degree zero and, consequently, it is simply a function from $\mathfrak{B}(D, X)$. Moreover, the relation $sx = (\alpha \oplus \bar{\partial})\psi$ is equivalent to $x = (z - a)\psi(z)$, $z \in D$ and $(\partial/\partial\bar{z})\psi = 0$. Therefore ψ is an analytic function on D and, consequently, we have, in this case, $\mathrm{sp}(a, x) = \sigma(a, x)$.

If the equation in ψ, $sx = (\alpha \oplus \bar{\partial})\psi$, has a global solution defined on $r(a, x)$ then, for any open neighbourhood V of $\mathrm{sp}(a, x)$, the cohomology class of sx with respect to $\alpha \oplus \bar{\partial}$ contains a form χ with compact support in V [**4**, §1] and, consequently, we get the following local variant of (3):

$$x = \frac{1}{(2\pi i)^n} \int_V (-1)^n \pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n. \tag{3'}$$

Just this local variant of (3) motivates our Definition 1. It is easy to see that, in the case $n = 1$, the equation $sx = (\alpha \oplus \bar{\partial})\psi$ has a global solution if and only if the operator a has the single-valued extension property [**3**], [**2**].

DEFINITION 2. *The system a is said to have the localization property (in short, property* L) *if $H^{n-1}(\mathfrak{B}(G, X), \alpha \oplus \bar{\partial}) = 0$ for any open set $G \subset C^n$.*

We have denoted by $H^{n-1}(\mathfrak{B}(G, X), \alpha \oplus \bar{\partial})$ the cohomology module of order $n - 1$ for the cochain complex consisting of the spaces $\Lambda^p[\sigma \cup d\bar{z}, \mathfrak{B}(G, X)]$ and of $\alpha \oplus \bar{\partial}$ as coboundary operator.

Let U be an open set in C^n and let us denote by $\mathfrak{A}(U, X)$ the space of all X-valued analytic functions on U. Denote by α the operator defined by

$$(\alpha\psi)(z) = \left[(z_1 - a_1)s_1 + \cdots + (z_n - a_n)s_n\right] \wedge \psi(z), \qquad z \in U,$$

where ψ is an exterior form in s with coefficients in $\mathfrak{A}(U, X)$. Then the object consisting of the spaces $\Lambda^p[\sigma, \mathfrak{A}(U, X)]$ and of α as coboundary operator is a cochain complex. We shall denote by $H^i(\mathfrak{A}(U, X), \alpha)$ the cohomology modules of this complex.

DEFINITION 3. *The system a is said to have the analytic localization property (in short property* LA) *if $H^i(\mathfrak{A}(U, X), \alpha) = 0$, $0 \leqslant i \leqslant n - 1$, for any open polydisc $U \subset C^n$.*

We have proved in [**4**] that property LA implies property L. If $n = 1$ then the properties are equivalent to the single-valued extension property.

DEFINITION 4 [**7**]. *Let $K \subset C^n$ be a compact set. The spectral hull of K is the set of all elements $w = (w_1, \ldots, w_n) \in C^n$ with the property that the equation*

$$(z_1 - w_1)f_1(z) + \cdots + (z_n - w_n)f_n(z) = 1$$

has no analytic solutions $f_1, \ldots, f_n$ in any open neighbourhood of K.

We shall prove that, for any $x \in X$, $\sigma(a, x)$ is contained in the spectral hull of $\mathrm{sp}(a, x)$. For that we need two lemmas.

LEMMA 1. *Let K be a compact subset of C^n. If $w = (w_1, \ldots, w_n)$ does not belong to the spectral hull of K then there exist an open neighbourhood U of K,*

an open neighbourhood V of w and n scalar functions $f_1(z, \zeta), \ldots, f_n(z, \zeta)$ analytic on $U \times V$, satisfying

$$\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z, \zeta) \equiv 1, \qquad (z, \zeta) \in U \times V.$$

PROOF. Since w does not belong to the spectral hull of K, it follows, according to Definition 4, that there exist an open neighbourhood U_1 of K and n scalar analytic functions $f_1, \ldots, f_n$ on U_1, such that

$$\sum_{j=1}^{n} (z_j - w_j) f_j(z) \equiv 1, \qquad z \in U_1.$$

From here we obtain

$$\begin{aligned}\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z) &= \sum_{j=1}^{n} (z_j - w_j) f_j(z) + \sum_{j=1}^{n} (w_j - \zeta_j) f_j(z) \\ &= 1 + \sum_{j=1}^{n} (w_j - \zeta_j) f_j(z), \qquad z \in U_1.\end{aligned}$$

Let U be an open relatively compact neighbourhood of K such that $\overline{U} \subset U_1$. It is easy to see that there exists an open neighbourhood V of w such that $|\Sigma_{j=1}^{n}(w_j - \zeta_j) f_j(z)| < 1$ for any $\zeta \in V$ and any $z \in U$. Thus, for any $(z, \zeta) \in U \times V$, the complex number

$$\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z) = 1 + \sum_{j=1}^{n} (w_j - \zeta_j) f_j(z)$$

is different from zero. Denoting

$$h(z, \zeta) = \left[\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z) \right]^{-1}, \qquad (z, \zeta) \in U \times V,$$

we shall obtain an analytic function on $U \times V$ satisfying

$$h(z, \zeta) \left[\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z) \right] \equiv 1, \qquad (z, \zeta) \in U \times V.$$

Therefore we may define

$$f_j(z, \zeta) = h(z, \zeta) f_j(z), \qquad (z, \zeta) \in U \times V, 1 \leqslant j \leqslant n,$$

and the statement of Lemma 1 follows.

LEMMA 2. *Suppose that the system a satisfies* L. *Let U be an open neighbourhood of the set* $\mathrm{sp}(a, x)$ *and f a scalar analytic function on U. Then, for any form appearing in* (3′) *and any number j, $1 \leqslant j \leqslant n$, we have*

$$\int_U (-1)^n f(z)(z_j - a_j) \pi \chi \wedge dz_1 \wedge \cdots \wedge dz_n = 0.$$

PROOF. We shall apply the theorem of Stokes. Let j be a fixed number, $1 \leqslant j \leqslant n$, and let us denote by $\tilde{a}_j$ the operator defined by $(\tilde{a}_j \psi)(z) = (z_j - a_j)\psi(z)$, $z \in U$, where ψ is a form in s and $d\bar{z}$ having coefficients in $\mathcal{B}(U, X)$.

Let χ be an arbitrary form from those appearing in (3′). Then there exists a form ψ^* on U such that $sx - \chi = (\alpha \oplus \bar{\partial})\psi^*$. By applying the operator $\tilde{a}_j$ to both sides of this equality and multiplying by f, we obtain

$$sf\tilde{a}_j x - f\tilde{a}_j\chi = (\alpha \oplus \bar{\partial})(f\tilde{a}_j\psi^*).$$

On the other hand, denoting

$$\tilde{\psi}_j(z) = f(z)xs_1 \wedge \cdots \wedge \hat{s}_j \wedge \cdots \wedge s_n, \qquad z \in U,$$

we have $sf\tilde{a}_j x = (\alpha \oplus \bar{\partial})\tilde{\psi}_j$, whence

$$f\tilde{a}_j\chi = (\alpha \oplus \bar{\partial})(\tilde{\psi}_j - f\tilde{a}_j\psi^*).$$

Taking into account that χ has compact support in U, it follows that $f\tilde{a}_j\chi$ has compact support in U. By using property L we shall prove that $f\tilde{a}_j\chi = (\alpha \oplus \bar{\partial})\psi_j$ on U, where ψ_j has also compact support. With this result at hand, we get

$$f\tilde{a}_j\pi\chi = \pi(f\tilde{a}_j\chi) = \pi(\alpha \oplus \bar{\partial})\psi_j = \bar{\partial}(\pi\psi_j).$$

From here we obtain

$$\begin{aligned}
&\int_U (-1)^n f(z)(z_j - a_j)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\quad = \int_U (-1)^n f\tilde{a}_j\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\quad = \int_U (-1)^n \bar{\partial}(\pi\psi_j) \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\quad = \int_U (-1)^n d(\pi\psi_j \wedge dz_1 \wedge \cdots \wedge dz_n).
\end{aligned}$$

Taking into account that ψ_j has compact support in U and applying the theorem of Stokes, we deduce that the last integral is equal to zero and, consequently,

$$\int_U (-1)^n f(z)(z_j - a_j)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n = 0.$$

Therefore it remains to prove that there exists a form ψ_j with compact support such that $f\tilde{a}_j\chi = (\alpha \oplus \bar{\partial})\psi_j$. Denote by K the support of χ (which is a compact set). Since $f\tilde{a}_j\chi = (\alpha \oplus \bar{\partial})(\tilde{\psi}_j - f\tilde{a}_j\psi^*)$ and $\chi = 0$ on $U \setminus K$, we have $(\alpha \oplus \bar{\partial})(\tilde{\psi}_j - f\tilde{a}_j\psi^*) = 0$ on $U \setminus K$. Thus, by applying property L for $G = U \setminus K$, there exists a form φ_j such that $\tilde{\psi}_j - f\tilde{a}_j\psi^* = (\alpha \oplus \bar{\partial})\varphi_j$. Consider now two open relatively compact sets U_1 and U_2 such that $K \subset U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$. Let h be a C^∞ scalar function on U such that $h = 0$ on U_1 and $h = 1$ on $U \setminus U_2$. Let us define the form $\tilde{\varphi}_j$ by

$$\begin{aligned}
\tilde{\varphi}_j &= h\varphi_j \quad \text{on } U \setminus K, \\
&= 0 \quad \text{on } U_1.
\end{aligned}$$

Denoting $\psi_j = (\tilde{\psi}_j - f\tilde{a}_j\psi^*) - (\alpha \oplus \bar{\partial})\tilde{\varphi}_j$, we obtain

$$(\alpha \oplus \bar{\partial})\psi_j = (\alpha \oplus \bar{\partial})(\tilde{\psi}_j - f\tilde{a}_j\psi^*) = f\tilde{a}_j\chi.$$

On the other hand, we have $\tilde{\varphi}_j = h\varphi_j$ on $U \setminus K$ and $h = 1$ on $U \setminus U_2$, whence $\psi_j = 0$ on $U \setminus U_2$. This completes the proof.

We can prove now our main result concerning the relation between the spectrum $\sigma(a, x)$ defined in the Introduction and the spectrum $\mathrm{sp}(a, x)$.

THEOREM. *Let a be a commuting system of operators satisfying property* L. *Then, for any $x \in X$, the analytic spectrum $\sigma(a, x)$ is contained in the spectral hull of the spectrum* $\mathrm{sp}(a, x)$.

PROOF. We shall prove that if w does not belong to the spectral hull of $\mathrm{sp}(a, x)$ then $w \in \rho(a, x)$, from where the Theorem will obviously follow. According to Lemma 1, there exist an open neighbourhood U of $\mathrm{sp}(a, x)$, an open neighbourhood V of w and n scalar functions $f_1(z, \zeta), \ldots, f_n(z, \zeta)$, analytic on $U \times V$ and satisfying

$$\sum_{j=1}^{n} (z_j - \zeta_j) f_j(z, \zeta) \equiv 1, \qquad (z, \zeta) \in U \times V.$$

Let us define the X-valued functions f_j on V by

$$f_j(\zeta) = -\frac{1}{(2\pi i)^n} \int_U (-1)^n f_j(z, \zeta)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n, \quad 1 \leqslant j \leqslant n,$$

where χ is one of the forms appearing in (3′). Since the form χ has compact support, the forms appearing under the integral also have compact support. Moreover, since $f_j(z, \zeta)$ are analytic functions, $1 \leqslant j \leqslant n$, it is easy to see that f_j are analytic functions on V, $1 \leqslant j \leqslant n$. We shall prove that $\sum_{j=1}^{n}(\zeta_j - a_j)f_j(\zeta) \equiv x$, $\zeta \in V$, and thus it will follow that $w \in \rho(a, x)$, as desired.

By applying Lemma 2 for the functions $f_j(z, \zeta)$, we get

$$\int_U (-1)^n f_j(z, \zeta)(z_j - a_j)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n = 0, \qquad 1 \leqslant j \leqslant n.$$

Consequently, we have

$$\begin{aligned}
&\sum_{j=1}^{n} (\zeta_j - a_j) f_j(\zeta) \\
&\quad = -\frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (\zeta_j - a_j) f_j(z, \zeta)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\quad = -\frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (\zeta_j - z_j) f_j(z, \zeta)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\qquad - \frac{1}{(2\pi i)^n} \int_U (-1)^n \sum_{j=1}^{n} (z_j - a_j) f_j(z, \zeta)\pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n \\
&\quad = \frac{1}{(2\pi i)^n} \int_U (-1)^n \pi\chi \wedge dz_1 \wedge \cdots \wedge dz_n = x.
\end{aligned}$$

For the last equality we have used (3′). Therefore

$$\sum_{j=1}^{n} (\zeta_j - a_j) f_j(\zeta) \equiv x, \qquad \zeta \in V,$$

and the proof is finished.

REMARK. It is possible to define a local spectrum similar to $\mathrm{sp}(a, x)$ by using the space C^∞ instead of the space $\mathfrak{B}$. It is clear that $\mathrm{sp}(a, x)$ will be contained in this spectrum. All the results proved in [4] for $\mathrm{sp}(a, x)$ also remain valid for this new spectrum. Indeed, the properties of the space $\mathfrak{B}$ used in [4] are still valid for the space C^∞ (see also [5]).

REFERENCES

1. E. Albrecht, *Funktionalkalküle in mehreren veränderlichen*, Ph.D. Dissertation, Mainz Univ., 1972.

2. I. Colojoară and C. Foiaş, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

3. N. Dunford, *Spectral operators*, Pacific J. Math. **4** (1954), 321–354.

4. Ş, Frunză, *The Taylor spectrum and spectral decompositions*, J. Functional Analysis **19** (1975), 390–421.

5. _____, *On Taylor functional calculus*, Studia Math. **52** (1974), 239–242.

6. J. L. Taylor, *A joint spectrum for several commuting operators*, J. Functional Analysis **6** (1970), 172–191.

7. _____, *Analytic functional calculus for several commuting operators*, Acta Math. **125** (1970), 1–38.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IAŞI, IAŞI, ROMÂNIA