LOCAL COMPACTNESS AND HEWITT REALCOMPACTIFICATIONS OF PRODUCTS

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Abstract. In this note we prove McArthur's conjecture [6]: If card X is nonmeasurable and if v(X X Y) = vX X vY holds for each space Y, then X is locally compact. Consequently, we can completely characterize the class of all spaces X such that for each space Y, v(X X Y) = vX X vY holds.

1. Introduction. All spaces considered in this note will be completely regular Hausdorff. For a space X, vX denotes the Hewitt realcompactification of X, and the symbolism v(X X Y) = vX X vY means that X X Y is C-embedded in vX X vY. Following [6], let R denote the class of all spaces X such that for each space Y, v(X X Y) = vX X vY holds. It is known that a locally compact realcompact space of nonmeasurable cardinal is a member of R and that every member of R is realcompact (Comfort [1, Corollary 2.2], McArthur [6, Theorem 5.2]). In [6], McArthur conjectured that if card X is nonmeasurable and X is a member of R, then X is locally compact. The main purpose of this note is to establish his conjecture positively. More precisely, we can prove the following theorems. The implication (a) → (b) of Theorem 1 was proved by Comfort [1].

Theorem 1. For a space X of nonmeasurable cardinal the following conditions are equivalent:

(a) X is locally compact.
(b) X X Y is C-embedded in X X vY for each space Y.

Theorem 2. For a space X of nonmeasurable cardinal the following conditions are equivalent:

(a) X is locally pseudocompact.
(b) X X Y is C-embedded in X X vY for each k-space Y.

We remark that, in Theorems 1 and 2, the assumption "card X is nonmeasurable" is useful only for the implication (a) → (b). Combining these theorems with the results of Comfort and McArthur, quoted above, and Hušek [4, Theorem 3], we have the following theorems.

Theorem 3. For a space X the following conditions are equivalent:
(a) $X$ is locally compact, realcompact and card $X$ is nonmeasurable.
(b) $\nu(X \times Y) = \nu X \times \nu Y$ holds for each space $Y$.

**THEOREM 4.** For a space $X$ the following conditions are equivalent:
(a) $\nu X$ is locally compact and card $X$ is nonmeasurable.
(b) $\nu(X \times Y) = \nu X \times \nu Y$ holds for each $k$-space $Y$.

For the notions of locally pseudocompact spaces and $k$-spaces see [1]. For an ordinal $\alpha$, we denote by $W(\alpha)$ the set of all ordinals less than $\alpha$ topologized with order topology, and by $\omega_0$ the first infinite ordinal. Other terms can be found in [3].

2. Proofs of theorems.

PROOF OF THEOREM 1. (a) $\rightarrow$ (b). This is the result of Comfort [1, Theorem 2.1]. (b) $\rightarrow$ (a). Suppose, on the contrary, that $X$ is not locally compact at $x_0 \in X$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a neighborhood base at $x_0$ in $X$. Then, for each $\lambda \in \Lambda$, $cl_{\beta X} G_\lambda$ is not compact, and thus there exists a point $x_\lambda \in cl_{\beta X} G_\lambda \cap (\beta X - X)$, where $\beta X$ is the Stone-Čech compactification of $X$. For each $\lambda \in \Lambda$, let $\{G(\lambda, \sigma) \mid \sigma \in \Sigma_\lambda\}$ be a neighborhood base at $x_\lambda$ in $\beta X$. For each $\sigma \in \Sigma_\lambda$, we can choose a point $x(\lambda, \sigma) \in X$ and an open set $H(\lambda, \sigma)$ in $X$ such that $x(\lambda, \sigma) \in H(\lambda, \sigma) \subset G(\lambda, \sigma) \cap G_\lambda$. Let $s_\lambda$ be an ideal point, and set $S_\lambda = \Sigma_\lambda \cup \{s_\lambda\}$, topologized as follows: Each point of $\Sigma_\lambda$ is isolated and $\{J(\lambda, \sigma) \mid \sigma \in \Sigma_\lambda\}$ is a neighborhood base at $s_\lambda$, where $J(\lambda, \sigma) = \{s_\lambda\} \cup \{\tau \in \Sigma_\lambda \mid G(\lambda, \sigma) \supseteq G(\lambda, \tau)\}$. Let $n$ be a regular cardinal greater than $sup\{\text{card} \Sigma_\lambda \mid \lambda \in \Lambda\}$, and let $\omega_n$ be the initial ordinal of $n$. For each $\lambda \in \Lambda$, let

$$T_{\lambda(1)} = \{\{(\lambda(1), \gamma, \beta) \mid \gamma < \omega_n, \beta < \omega_0\}\}$$

be the copy of $W(\omega_n + 1) \times W(\omega_0 + 1)$, and let

$$T_{\lambda(2)} = \{\{(\lambda(2), \gamma, s) \mid \gamma < \omega_n, s \in S_\lambda\}\}$$

be the copy of $W(\omega_n + 1) \times S_\lambda$. By identifying a point $\{(\lambda(1), \gamma, \omega_0)\}$ with $\{(\lambda(2), \gamma, s)\}$ for $\gamma < \omega_n$, we have a quotient space $T_\lambda$ and a quotient map $f_\lambda$: $T_{\lambda(1)} \oplus T_{\lambda(2)} \rightarrow T_\lambda$, where $A \oplus B$ denotes the topological sum of $A$ and $B$. Let us set $Z = \bigoplus \{T_\lambda \mid \lambda \in \Lambda\}$, and let $Y_0$ be the quotient space obtained from $Z$ by collapsing a set $\{f_\lambda((\lambda(1), \omega_0, \beta)) \mid \lambda \in \Lambda\}$ to a single point $y(\beta) \in Y_0$ for $\beta < \omega_0$. Let $g: Z \rightarrow Y_0$ be the quotient map, and set $h_\lambda = g \circ f_\lambda$ for each $\lambda \in \Lambda$. Then $y(\omega_0) = h_\lambda((\lambda(2), \omega_n, s_\lambda))$ for each $\lambda \in \Lambda$. Let us set $Y = Y_0 - \{y_0\}$, where $y_0 = y(\omega_0)$. We shall now prove that $Y_0 \subset \nu Y$ by showing that $Y$ is $C$-embedded in $Y_0$. Let $\phi$ be a real-valued continuous function on $Y$. For each $\lambda \in \Lambda$, by the same argument as in [3, 8.20], there is $\gamma_\lambda \in W(\omega_n)$ such that $\theta_\lambda = \phi \circ (h_\lambda|h_\lambda^{-1}(Y))$ takes on the constant value $t_\lambda$ on $\{(\lambda(1), \gamma, \omega_0) \mid \gamma_\lambda \leq \gamma \leq \omega_0\} \cup \{(\lambda(2), \gamma, s) \mid \gamma \leq \gamma < \omega_n\}$. Since

$$\theta_\lambda((\lambda(1), \omega_n, \beta)) = \theta_\mu((\mu(1), \omega_n, \beta))$$

for $\lambda, \mu \in \Lambda$ and for each $\beta < \omega_0$, we have $t_\lambda = t_\mu$ for $\lambda, \mu \in \Lambda$. Extend $\phi$ over $Y_0$ by setting $\phi(y_0) = t_\lambda$. Then it is easy to see that the extension $\phi$ is
continuous. Thus $Y$ is $C$-embedded in $Y_0$, and hence $Y_0 \subset \nu Y$. It remains to show that $X \times Y$ is not $C$-embedded in $X \times \nu Y$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, let us set
\[ y(\lambda, \sigma) = h_\lambda((\lambda(2), \omega_\sigma, \sigma)), \]
\[ K(\lambda, \sigma) = h_\lambda\left(\{(\lambda(2), \gamma, \sigma)|\gamma < \omega_\sigma\}\right). \]
And let us set
\[ p(\lambda, \sigma) = (x(\lambda, \sigma), y(\lambda, \sigma)) \in X \times Y, \]
\[ L(\lambda, \sigma) = H(\lambda, \sigma) \times K(\lambda, \sigma) \subset X \times Y, \]
\[ \mathcal{L} = \{L(\lambda, \sigma)|\lambda \in \Lambda, \sigma \in \Sigma_\lambda\}. \]
Then $L(\lambda, \sigma)$ is a neighborhood at $p(\lambda, \sigma)$ in $X \times Y$. Now we show that $\mathcal{L}$ is discrete in $X \times Y$. To do this, let $p = (x, y) \in X \times Y$; then $y = h_\mu((\mu(i), \delta, t))$ for some $\mu \in \Lambda$, $i \in \{1, 2\}$, $\delta < \omega_\sigma$ and $t \in W(\omega_0 + 1) \oplus S_\mu$. If $t \in W(\omega_0 + 1)$ and $t < \omega_\sigma$, then
\[ V(y) = \bigcup \{h_\lambda(T_{\lambda(1)})|\lambda \in \Lambda\} \cap Y \]
is a neighborhood at $y$ in $Y$ such that $V(y) \cap K(\lambda, \sigma) = \emptyset$ for each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, and hence $X \times V(y)$ is a neighborhood at $p$ which meets no member of $\mathcal{L}$. If $t = \omega_0$ or $s_\mu$, then there exist $\tau \in \Sigma_\mu$ and a neighborhood $V(x)$ at $x$ such that $V(x) \cap G(\mu, \tau) = \emptyset$. If we set
\[ V(y) = \bigcup \{h_\mu((\mu(1), \gamma, \beta)|\gamma < \delta, \beta < \omega_\sigma\} \]
\[ \bigcup \{h_\mu((\mu(2), \gamma, s)|\gamma < \delta, s \in J(\mu, \tau)\}, \]
then $V(y)$ is a neighborhood at $y$ in $Y$ such that $V(x) \times V(y)$ meets no member of $\mathcal{L}$. If $t \in \Sigma_\mu$, then $X \times K(\mu, t)$ is a neighborhood at $p$ which meets only $L(\mu, t)$. Hence $\mathcal{L}$ is discrete in $X \times Y$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, there is a real-valued continuous function $\psi_{(\lambda, \sigma)}$ on $X \times Y$ such that $\psi_{(\lambda, \sigma)}(p(\lambda, \sigma)) = 0$ and $\psi_{(\lambda, \sigma)}(q) = 1$ for each $q \in (X \times Y) - L(\lambda, \sigma)$. If we define a function $\psi$ by
\[ \psi(q) = \inf\{\psi_{(\lambda, \sigma)}(q)|\lambda \in \Lambda, \sigma \in \Sigma_\lambda\}, \quad q \in X \times Y, \]
then $\psi$ is continuous, since $\mathcal{L}$ is discrete. For our purpose, it suffices to show that $\psi$ admits no continuous extension to the point $p_\sigma = (x_\sigma, y_\sigma) \in X \times \nu Y$. Let $U$ be a given neighborhood at $p_\sigma$. There exist $\mu \in \Lambda$ and a neighborhood $V(y_\sigma)$ at $y_\sigma$ in $Y_\sigma$ such that $p_\sigma \in G_\mu \times V(y_\sigma) \subset U$. Then $y(\mu, \tau) \in V(y_\sigma)$ for some $\tau \in \Sigma_\mu$, and hence $p(\mu, \tau) \in U$ and $\psi(p(\mu, \tau)) = 0$. On the other hand, $y(\beta)$ is in $V(y_\sigma)$ for some $\beta < \omega_\sigma$, and then $q = (x_\sigma, y(\beta)) \in U$ and $\psi(q) = 1$. This shows that $\psi$ does not extend continuously to $p_\sigma$. Hence the proof is completed.

Before proving Theorem 2, we prove the implication (a) $\rightarrow$ (b) of Theorem 4, which slightly improves a theorem of Comfort [1, Theorem 2.4]. We denote
by \(\mu X\) the topological completion of \(X\) (i.e., the completion of \(X\) with respect to its finest uniformity).

**Proof of Theorem 4.** (a) \(\Rightarrow\) (b). Assume that \(\nu X\) is locally compact and card \(X\) is nonmeasurable. Let \(Y\) be a \(k\)-space. Then, by [1, Theorem 2.1], \(\nu X \times Y\) is \(C\)-embedded in \(\nu X \times \nu Y\). Since \(\nu X\) is locally compact, by [5, Theorem 1.5], we have \(\nu X = \mu X\). Hence \(\mu(\nu X \times Y) = \mu X \times \mu Y\) holds by [5, Theorem 2.3], and so \(\nu X \times Y\) is \(C\)-embedded in \(\mu X \times Y\) (= \(\nu X \times Y\)). Thus we have \(\nu(X \times Y) = \nu X \times \nu Y\).

**Proof of Theorem 2.** (a) \(\Rightarrow\) (b). Let \(X\) be a locally pseudocompact space of nonmeasurable cardinal and let \(Y\) be a \(k\)-space. Now it suffices to show that for each pseudocompact subset \(S\) of \(X\), \(S \times Y\) is \(C\)-embedded in \(S \times \nu Y\). To see this, let \(S\) be a given pseudocompact subset of \(X\), then we have \(\nu S = \beta S\) by [3, 8A4]. Thus \(\nu(S \times Y) = \nu S \times \nu Y\) holds by Theorem 4, (a) \(\Rightarrow\) (b) proved above, and hence \(S \times Y\) is \(C\)-embedded in \(S \times \nu Y\). (b) \(\Rightarrow\) (a). Suppose on the contrary that \(X\) is not locally pseudocompact at \(x_0 \in X\). Let \(\{G_\lambda | \lambda \in \Lambda\}\) be a neighborhood base at \(x_0\). Then, for each \(\lambda \in \Lambda\), \(\text{cl}_X G_\lambda\) is not pseudocompact, and thus we can find a countable decreasing family \(\{G(\lambda, \sigma) | \sigma \in \Sigma_\lambda\}\) of open sets in \(X\) such that \(\bigcap \{\text{cl}_X G(\lambda, \sigma) | \sigma \in \Sigma_\lambda\} = \emptyset\) and \(G(\lambda, \sigma) \subset G_\lambda\) for each \(\sigma \in \Sigma_\lambda\). Let us set \(H(\lambda, \sigma) = G(\lambda, \sigma)\), and choose a point \(x(\lambda, \sigma) \in H(\lambda, \sigma)\). We construct \(Y_0\) and \(Y\) quite similarly to the proof of Theorem 1. Examining the process, one sees that then each \(S_\lambda\) is compact, and hence \(Z\) is locally compact. Since every quotient space and open subspace of a \(k\)-space is a \(k\)-space, \(Y\) is a \(k\)-space. Therefore, by pursuing the proof of Theorem 1, we have Theorem 2.

To prove the implication (b) \(\Rightarrow\) (a) of Theorems 3 and 4, we need a theorem of Husek [4, Theorem 3]. His theorem can be restated as follows:

**Husek's Theorem.** For a space \(X\) the following conditions are equivalent:

(a) card \(X\) is nonmeasurable.

(b) \(\nu(X \times Y) = \nu X \times \nu Y\) holds for each discrete space \(Y\).

**Proof of Theorem 3.** (a) \(\Rightarrow\) (b) is the result of Comfort quoted in the introduction. (b) \(\Rightarrow\) (a). By Husek's theorem, card \(X\) is nonmeasurable. It follows from Theorem 1 and [6, Theorem 5.2] that \(X\) is locally compact and realcompact.

**Proof of Theorem 4.** (b) \(\Rightarrow\) (a). Since a discrete space is a \(k\)-space, by Husek's Theorem, card \(X\) is nonmeasurable. By Theorem 2, \(\nu X\) is locally pseudocompact, and hence is locally compact, because every pseudocompact realcompact space is compact (cf. [3, 8E1]).

**3. Remarks.** (1) If \(\nu X\) is locally compact, then \(X\) is locally pseudocompact, but the converse is false (see [1]).

(2) The space \(Y\) constructed in the proof of Theorems 1 and 2 and [6, Theorem 5.2] is 0-dimensional (i.e., ind \(Y\) = 0). Hence all theorems in this note remain true if "for each \((k\)-) space \(Y\)" is replaced by "for each 0-dimensional \((k\)-) space \(Y\)".
(3) A space similar to the space $S_\chi$ in the proof of Theorem 1 was used in [6] to show that every member of $\mathfrak{B}$ is realcompact.

(4) A space $X$ is said to be **topologically complete** if it is complete with respect to its finest uniformity (i.e., $X = \mu(X)$). In [7], Morita proved that if $X$ is locally compact topologically complete, then $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space $Y$, and Isiwata [5] proved that if $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space $Y$, then $X$ is topologically complete (cf. also [8]). Hence the analogous results of Theorems 1-4 remain true, with no cardinality conditions, for topological completions (in this case, we need to use [5, Theorem 2.3], [7, Theorem 3.1] and [2, Lemma 3.1] instead of Theorem 4, (a) $\rightarrow$ (b), [3, 8A4] and [3, 8E1], respectively).

**Added in Proof.** Recently, Blair and Hager (*z*-embedding in $\beta X \times \beta Y$, *Set theoretic topology*, Academic Press, New York, 1977) asked whether the following condition (d') implies that $X \times Y$ is *z*-embedded in $\beta X \times \beta Y$ (i.e., each zero-set of $X \times Y$ is the trace on $X \times Y$ of a zero-set of $\beta X \times \beta Y$):

(d') For every real-valued continuous function $f$ on $X \times Y$ and every $\varepsilon > 0$, there is a countable open rectangular cover $\{G_n\}$ of $X \times Y$ such that $\sup \{|f(p) - f(q)| \mid p, q \in G_n\} < \varepsilon$ for each $n$.

In the same paper, they proved that if $X$ has a countable base, then $X \times Y$ satisfies (d') for each space $Y$, and that if $X \times Y$ is *z*-embedded in $\beta X \times \beta Y$, then $\nu(X \times Y) = \nu X \times \nu Y$ holds. From these facts, since there exists a nonlocally compact space with a countable base, Theorem 3 answers this question negatively. Furthermore, combining Theorem 3 with their results (3.2, 3.3), we obtain: $X$ is a locally compact space with a countable base if and only if $X \times Y$ is *z*-embedded in $\beta X \times \beta Y$ for each space $Y$.

**REFERENCES**


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