ON THE NOTION OF \( n \)-CARDINALITY

TEODOR C. PRZYMUSiNSKI

ABSTRACT. In this paper we introduce and investigate the notion of \( n \)-cardinality, which turned out to be useful in constructions involving product spaces and has a number of interesting applications.

In this paper we introduce and investigate the notion of \( n \)-cardinality, which turned out to be useful in constructions involving product spaces and has a number of interesting applications (see \([P]\), \([P_1]\), \([P_2]\), and \([P_3]\)).

The notion of \( n \)-cardinality arose after discussions with Eric van Douwen, who also proved an important case of Theorem 1 (Corollary 1). The author is grateful to him for his valuable suggestions.

Throughout this paper \( n \) denotes a natural number and \( c = 2^n \). Let \( X \) be an arbitrary set. For a point \( p = (p_1, \ldots, p_n) \) from \( X^n \) by \( p \) we shall denote the set \( \{p_1, \ldots, p_n\} \) of coordinates of \( p \). By \( p_i \) we shall always mean the \( i \)th coordinate of \( p \). For undefined notions and symbols the reader is referred to \([E]\).

Lemma 1. For a subset \( A \) of \( X^n \) the following cardinals are well defined and they are equal provided that one of them or—equivalently—all of them are infinite:

(i) \( \max \{|B|: B \subset A \text{ and } p_i \neq q_i, \text{ for } i = 1, 2, \ldots, n \text{ and every two distinct points } p \text{ and } q \text{ from } B \} \);

(ii) \( \max \{|B|: B \subset A \text{ and } p \cap q = \emptyset, \text{ for every two distinct points } p \text{ and } q \text{ from } B \} \);

(iii) \( \min \{|Y|: Y \subset X \text{ and } A \subset \bigcup_{i=1}^{n} (X^{i-1} \times Y \times X^{n-i}) \} \).

Proof. Let us denote by \( \tau \) the cardinal number defined in (iii). Since \( \tau \) is well defined it suffices to show that: (a) if \( \tau \geq \omega \), then cardinals described in (i) and (ii) coincide with \( \tau \) and (b) if \( \tau < \omega \), then cardinals described in (i) and (ii) are finite.

Let us note first that if \( B \) is a subset of \( A \) such that \( p_i \neq q_i, \text{ for } i = 1, 2, \ldots, n \text{ and every two distinct points } p \text{ and } q \text{ from } B \), then \( |B| \leq n \cdot \tau \).

Indeed, let \( Y \) be a subset of \( X \) of cardinality \( \tau \) such that \( A \subset \bigcup_{i=1}^{n} (X^{i-1} \times Y \times X^{n-i}) \). For every \( i = 1, 2, \ldots, n \) and every \( y \in Y \) there exists at most one \( p \in B \) such that \( p_i = y \), therefore \( |B| \leq n \cdot \tau \). From this fact we deduce (b) and infer that in order to prove (a) it suffices to construct a subset \( B \) of \( A \)
of cardinality \( \tau \) such that \( \hat{p} \cap \hat{q} = \emptyset \), for every two distinct points \( p, q \in B \).

Assume that \( \tau \geq \omega \). We shall construct points \( p(\alpha) \) of \( B \), for \( \alpha < \tau \), by transfinite recursion. Assume that points \( p(\beta) \in A \) have been constructed for \( \beta < \alpha \) so that \( \hat{p}(\beta) \cap \hat{p}(\gamma) = \emptyset \), if \( \beta \neq \gamma \).

The set \( Z = \bigcup \{ \hat{p}(\beta) : \beta < \alpha \} \) has cardinality \( < \tau \) and therefore there exists a point

\[
p(\alpha) \in A \setminus \bigcup_{i=1}^{n} \left( X_i \times Z \times X_{n-i} \right).
\]

Clearly \( \hat{p}(\alpha) \cap \hat{p}(\beta) = \emptyset \), for every \( \beta < \alpha \), which completes the proof of the Lemma.

**Definition 1.** For a subset \( A \) of \( X^n \), where \( X \) is an arbitrary set, we define the \( n \)-cardinality \( |A|_n \) of \( A \) (with respect to \( X^n \)) by \( |A|_n = \max \{|B| : B \subset A \) and \( p_i \neq q_i \), for every two distinct points \( p \) and \( q \) from \( B \) and \( i = 1, 2, \ldots, n \} \). We say that \( A \) is \( n \)-countable (\( n \)-uncountable) if \( |A|_n < \omega \) (\( |A|_n \geq \omega \)).

It follows from Lemma 1 that \( n \)-cardinality is well defined and moreover:

1. \( |A|_1 = |A| \); i.e. \( n \)-cardinality generalizes the notion of cardinality;
2. \( |A|_n \leq |A| \);
3. \( |A|_2 = \min \{|Y| : A \subset Y \times X \cup X \times Y \} \),

provided that \( |A|_2 \) is infinite.

**Remark 1.** We can analogously define the \( n \)-cardinality of a subset \( A \) of \( \prod_{i=1}^{n} X_i \), where \( X_i \)'s are arbitrary sets, however, this potentially more general definition can be reduced to the previous one by observing that the so defined \( n \)-cardinality coincides with the \( n \)-cardinality of \( A \) with respect to \( X^n \), where \( X = \bigoplus_{i=1}^{n} X_i \). Making use of this observation, one can easily show that all results proved in this paper for subsets of \( X^n \) are actually valid--after obvious modifications--for subsets of the products \( \prod_{i=1}^{n} X_i \).

The following theorem generalizes a result of van Douwen (see Corollary 1).

**Theorem 1 (Main).** Let \( X \) be a complete separable metric space and \( B \) a Borel subset of \( X^n \). The following statements are equivalent:

i. \( B \) is \( n \)-uncountable;

ii. \( B \) has \( n \)-cardinality continuum;

iii. \( B \) contains a homeomorphic image \( h(C) \) of the Cantor set \( C \) such that \( \hat{h}(x) \cap \hat{h}(y) = \emptyset \), for \( x \neq y \);

iv. \( B \) contains a homeomorphic image \( h(C) \) of the Cantor set \( C \) by the diagonal

\[
h = \bigtriangleup_{i=1}^{n} h_i : C \to X^n
\]

of homeomorphic embeddings \( h_i : C \to X \);

v. (for \( n > 1 \)) \( B \) contains the graph of a homeomorphic embedding \( h \):
ON THE NOTION of \( n \)-CARDINALITY

\[ C \to X^{n-1} \] of a Cantor subset \( C \) of \( X \) into \( X^{n-1} \) such that \( \hat{h}(x) \cap \hat{h}(y) = \emptyset \), for \( x \neq y \);

(vi) (for \( n > 1 \)) \( B \) contains the graph of the diagonal \( h = \bigtriangleup_{i=2}^{\infty} h_i; C \to X^{n-1} \) of homeomorphic embeddings \( h_i; C \to X \) of a Cantor subset \( C \) of \( X \) into \( X \).

**Proof.** It follows immediately from Lemma 1 that either of the conditions (ii)–(vi) implies (i). We shall show the converse.

It is known that every Borel subset of a separable complete metric space is a continuous image of the space \( P \) of irrationals (cf. [K, Theorem 1, Chapter III, §37]). Let \( f: P \to B \) be a continuous mapping of \( P \) onto \( B \) and assume that \( |B|^n > \omega \). Let us choose an arbitrary complete metric on \( P \). By Lemma 1 there exists a collection \( \{ p(s) \}_{s \in S} \) of points \( B \) such that \( \hat{p}(s) \cap \hat{p}(s') = \emptyset \), for \( s \neq s' \) and \( |S| = \omega_1 \).

For each \( s \in S \) choose an \( x_s \in f^{-1}(p(s)) \) and put \( T = \{ x_s \}_{s \in S} \). Without loss of generality we can assume that \( T \) is dense-in-itself (otherwise, since \( T \) is second countable, by the Bernstein Theorem we would remove countably many points from \( S \) and \( T \)).

For each \( m = 1, 2, \ldots \) and every sequence \( (d_1, \ldots, d_m) \), where \( d_i = 0 \) or \( 1 \), we will define a point \( t(d_1, \ldots, d_m) \in T \) and a closed ball \( B(d_1, \ldots, d_m) \) in \( P \) with the center at the point \( t(d_1, \ldots, d_m) \) and radius \( < 1/m \) so that:

\[
\begin{align*}
(4)_m & \quad B(d_1, \ldots, d_m) \subset B(d_1, \ldots, d_{m-1}), & \text{for } m > 1; \\
(5)_m & \quad \text{for each pair } (d_1, \ldots, d_m) \text{ and } (d'_1, \ldots, d'_m) \text{ of distinct sequences there exist disjoint subsets } V_0 \text{ and } V_1 \text{ of } X \text{ such that } f(B(d_1, \ldots, d_m)) \subset V'_0 \text{ and } f(B(d'_1, \ldots, d'_m)) \subset V'_1.
\end{align*}
\]

Let \( m = 1 \) and choose two distinct points \( t(0) \) and \( t(1) \) from \( T \). Since \( X \) is Hausdorff, there exist disjoint open subsets \( V_0 \) and \( V_1 \) of \( X \) with \( \hat{f}(t(j)) \subset V_j \), for \( j = 0, 1 \). By the continuity of \( f \) there exist closed balls \( B(0) \) and \( B(1) \) with centers at \( t(0) \) and \( t(1) \), respectively, and radii \( < 1 \) such that \( f(B(j)) \subset V'_j \), for \( j = 0, 1 \), which completes the first step of the inductive construction.

Assume that \( m > 2 \) and that an inductive step has been made for \( m - 1 \). Let us take an arbitrary sequence \( (d_1, \ldots, d_{m-1}) \) and find two distinct points \( t_j = t(d_1, \ldots, d_{m-1}, j), j = 0, 1 \), from \( T \) belonging to the interior of \( B(d_1, \ldots, d_{m-1}) \). Such points exist because \( T \) is dense-in-itself. We can find two disjoint open subsets \( V_j, j = 0, 1 \), of \( X \) such that

\[
\hat{f}(t(d_1, \ldots, d_{m-1}, j)) \subset V_j, \quad j = 0, 1.
\]

There exist closed balls \( B(d_1, \ldots, d_{m-1}, j), j = 0, 1 \), with centers at the points \( t_j \) and radii \( < 1/m \) such that

\[
B(d_1, \ldots, d_{m-1}, j) \subset B(d_1, \ldots, d_{m-1})
\]

and

\[
f(B(d_1, \ldots, d_{m-1}, j)) \subset V'_j, \quad j = 0, 1.
\]
It is easy to see that the conditions \((4)_m\) and \((5)_m\) are satisfied, which completes our inductive construction.

One easily sees that the subset

\[
C = \bigcap_{m=1}^{\infty} \left( \bigcup_{(d_1, \ldots, d_m)} \{ B(d_1, \ldots, d_m) : (d_1, \ldots, d_m) \in \{0, 1\}^m \} \right)
\]

of \(P\) is homeomorphic to the Cantor set (cf. [K, Chapter III, §36,1]) and that the continuous mapping \(h = f|C : C \to B \subset X^n\) has the property

\[
\hat{h}(x) \cap \hat{h}(y) = \emptyset, \quad \text{for } x \neq y,
\]
in particular, \(h\) is one-to-one. As a one-to-one continuous mapping into a Hausdorff space defined on a compact space \(C\), the mapping \(h\) is a homeomorphic embedding. Therefore, (iii) is satisfied and consequently, by Lemma 1, also (ii) follows.

Let \(h(x) = (h_1(x), \ldots, h_n(x))\), for \(x \in C\). Since the mappings \(h_i\) are continuous and one-to-one, they are homeomorphic embeddings and (iv) holds.

Assume that \(n > 1\) and let \(C^1 = h_1(C) \subset X\) and \(h_i^1 = h_i \circ h_1^{-1} : C^1 \to X\), for \(i = 2, 3, \ldots, n\). Clearly \(C^1\) is homeomorphic to the Cantor set, \(h_i^1\)'s are homeomorphic embeddings of \(C^1 \subset X\) into \(X\) and the graph of the diagonal \(h^1 = \triangle_{i=2}^n h_i^1 : C^1 \to X^{n-1}\) coincides with \(h(C)\). This shows that also conditions (v) and (vi) are satisfied and completes the proof. □

**Remark 2.** It follows from the above proof that conditions (i)-(vi) are actually equivalent for every analytic subset \(B\) of \(X^n\), where \(X\) is an arbitrary Hausdorff space (analytic sets are continuous images of irrationals). □

**Remark 3.** R. Pol pointed out that Theorem 1 (and also Theorem 3) can be derived from the results obtained recently by K. Kuratowski [K4, Corollary 3], however, the direct proof of these theorems seems to be simpler. □

The following corollary has been first proved by van Douwen [vD].

**Corollary 1.** Let \(X\) be a separable complete metric space. A closed subset \(F\) of \(X^n\) is either \(n\)-countable or has \(n\)-cardinality continuum. □

**Corollary 2.** Let \(X\) be a separable complete metric space. A Borel subset \(B\) of \(X^2\) is either contained in \((X \times A) \cup (A \times X)\), with \(A\) countable, or it contains a graph of a homeomorphic embedding \(h : C \to X\) of a Cantor subset \(C\) of \(X\) into \(X\). □

**Corollary 3 (ALEXANDROV-HAUSDORFF).** Every uncountable Borel subset of a separable complete metric space contains a Cantor set \(C\) and therefore, has cardinality continuum. □

The next theorem (and its corollary) generalizes the classical theorem of Bernstein (cf. [K, Theorem 1, §40, I]) on the existence of totally imperfect subsets of the real line and plays an important role in applications of \(n\)-cardinality (see [P], [P1], [P2], and [P3]).

**Theorem 2.** Let \(X\) be a separable complete metric space. There exist disjoint
subsets $A_i$ of $X$, where $i < \omega$, such that for every $n < \omega$, every $n$-uncountable Borel subset $B$ of $X^n$ and every $i < \omega$ we have

$$|B \cap A_i^n|_n = 2^\omega.$$  

**Proof.** Let us denote by $\mathcal{B}_n$ the family of all $n$-uncountable Borel subsets of $X^n$. Since there are at most continuum Borel subsets in a separable metric space, the cardinality of $\mathcal{B}_n$ is $\leq c$. Let $(B_a)_{a < c}$ be such an enumeration of all elements of $\mathcal{B} = \bigcup_{a < c} \mathcal{B}_a$ that every element from $\mathcal{B}$ is listed continuum many times. For each $\alpha < c$ there exists exactly one $n(\alpha)$ such that $B_a \in \mathcal{B}_{n(\alpha)}$.

For $\alpha < c$ and $i < \omega$ we will construct points $p(\alpha, i)$ belonging to $B_a$ in such a way that

$$\hat{p}(\alpha, i) \cap \hat{p}(\alpha', i') = \emptyset, \text{ if } (\alpha, i) \neq (\alpha', i').$$  

Let $p(0, i), i < \omega$, be arbitrary points from $B_0$ such that $\hat{p}(0, i) \cap \hat{p}(0, i') = \emptyset$, if $i \neq i'$. Such points exist because $B_0$ is $n(0)$-uncountable. Let us take $\alpha < c$ and assume that we have already constructed points $p(\beta, i)$, for $\beta < \alpha$ and $i < \omega$. The set $Y = \bigcup \{\hat{p}(\beta, i) \colon \beta < \alpha, i < \omega \}$ has cardinality less than $c$ and therefore by Theorem 1 the set

$$B_a^* = B_a \setminus \bigcup_{j=1}^{n} (X_j^{-1} \times Y \times X_j^{n-j}),$$  

where $n = n(\alpha)$, has $n$-cardinality continuum and consequently we can find for $i < \omega$ points $p(\alpha, i) \in B_a^*$, such that $\hat{p}(\alpha, i) \cap \hat{p}(\alpha, i') = \emptyset$, if $i \neq i'$ which completes the inductive construction. It is easy to see that (6) is satisfied.

Let us put $A_i = \bigcup_{a < c} \{\hat{p}(\alpha, i)\}$. Clearly the sets $A_i$, $i < \omega$, are disjoint. If $n < \omega$ and $B$ is an $n$-uncountable Borel subset of $X^n$ then there exist continuum many ordinals $\alpha < c$ such that $B = B_a$ and for every such $\alpha$ and every $i < \omega$ we have

$$p(\alpha, i) \in B_a \cap (\hat{p}(\alpha, i))^n \subset B \cap A_i^n.$$  

It follows from Lemma 1 and (6) that $|B \cap A_i^n|_n = 2^\omega$. □

**Corollary 4.** Let $X$ be a separable complete metric space. There exists a subset $A$ of $X$ such that for every $n < \omega$, the complement of any Borel subset of $X^n$ containing either $A^n$ or $(X \setminus A)^n$ is $n$-countable.

**Proof.** Let $A_i$'s be as in Theorem 2. Put $A = A_0$ and recall that the complement of a Borel set is a Borel set. □

The following theorem can be proved in a similar way as Theorem 1 using the Theorem of Arhangel'skiï [A].

**Theorem 3.** Let $X$ be a first countable complete Lindelöf space. A closed subset $F$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum. □

**Corollary 5.** Let $X$ be a first countable compact space. A closed subset $F$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum. □
Corollary 6 (Čech-Pospíšil-Arhangel’skiï). A first countable compact space is either countable or has cardinality continuum. □

Remark 4. Theorem 3 can be generalized in the following way: Let $X$ be a first countable Hausdorff space. A complete Lindelöf subspace $A$ of $X^n$ is either $n$-countable or has $n$-cardinality continuum. □

References


Instytut Matematyczny PAN, Sniadeckich 8, 00-950 Warsaw, Poland