STONE'S THEOREM
AND SPECTRAL SUBSPACES OF AUTOMORPHISMS

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Abstract. It is shown that spectral subspaces of automorphisms of a von Neumann algebra can be defined by use of Stone's theorem on unitary representations.

It is well known that the theory of spectral subspaces of automorphisms as developed by Arveson [1] generalizes Stone's theorem for unitary representations. In this note we shall show a converse result, thus indicating how the theory of spectral subspaces of abelian groups of automorphisms of von Neumann algebras, see [1], [2], [3], can be developed from Stone's theorem. The idea is that a *-automorphism of the bounded operators $B(H)$ on a Hilbert space $H$ is implemented by a unitary operator, and so restricts to an isometry of the Hilbert-Schmidt operators $H_2$ on $H$. Thus Stone's theorem can be used on unitary representations on $H_2$ and then "lifted" to $B(H)$.

Throughout this note we let $G$ be a locally compact abelian group with Haar measure $dt$ and dual group $\Gamma$; $t \mapsto u_t$ is a strongly continuous unitary representation of $G$ on the Hilbert space $H$, and $\alpha_t = \text{Ad}(u_t)$. Then $t \mapsto \alpha_t$ is a continuous representation of $G$ into the automorphism group $\text{Aut} B(H)$ of $B(H)$, i.e., $t \mapsto \varphi(\alpha_t(x))$ is a continuous function for all $x \in B(H)$, $\varphi \in B(H)_u$. Recall from [1], [2], [3] that if $f \in L^1(G)$ and $x \in B(H)$ then

$$\pi_\alpha(f)(x) = \int_G f(t)\alpha_t(x) \, dt, \quad Z(f) = \{ \gamma \in \Gamma : \hat{f}(\gamma) = 0 \},$$

$$\text{Sp}_\alpha(x) = \bigcap \{ Z(f) : f \in L^1(G), \pi_\alpha(f)(x) = 0 \}.$$ 

We assume $M$ is a von Neumann algebra acting on $H$ such that $\alpha_t(M) = M$, $t \in G$. Then if $E$ is a closed subset of $\Gamma$, its spectral subspace is

$$M^\alpha(E) = \{ x \in M : \text{Sp}_\alpha(x) \subset E \}.$$ 

Moreover, the subspaces $M^\alpha(E)$ determine $\alpha$ [1].

Lemma. Denote by $\tilde{\alpha}$, the restriction of $\alpha_t$ to the Hilbert-Schmidt operators $H_2$ on $H$. Then $t \mapsto \tilde{\alpha}_t$ is a weakly, hence strongly, continuous unitary representation of $G$ on $H_2$.

Proof. Let $x, y \in H_2$ and $\epsilon > 0$. Let $y = y_1 + y_2$ with $y_1$ of finite rank and
\[ \|y_2\|_2 < \epsilon \|x\|_2, \text{ where } \|z\|_2 = \langle z, z \rangle^{1/2}, \text{ and } \langle \cdot , \cdot \rangle \text{ is the inner product on } H_2. \]

Since \( \alpha_t(x) \to x \) ultraweakly as \( t \to e \), the identity in \( G \), there is a neighborhood \( N \) of \( e \) in \( G \) such that \( |\langle \alpha_t(x) - x, y_1 \rangle| < \epsilon \) for \( t \in N \). Then

\[
|\langle \alpha_t(x) - x, y \rangle| < |\langle \alpha_t(x) - x, y_1 \rangle| + |\langle \alpha_t(x) - x, y_2 \rangle| < \epsilon + 2\|x\|_2\|y_2\|_2 < 3\epsilon,
\]

proving the lemma.

By Stone's theorem applied to the continuous unitary representation \( t \to \alpha_t \) on \( H_2 \), there exists a projection valued measure \( P_\lambda \) on \( \Gamma \) with values in \( B(H_2) \) such that

\[ \alpha_t = \int \lambda(t) \, dP_\lambda. \]

If \( E \) and \( F \) are closed subsets of \( \Gamma \) we denote by \( E + F \) the closure of the set \( \{ \gamma + \lambda : \gamma \in E, \lambda \in F \} \). We denote by \( P(F) \) the closed subspace of \( H_2 \) obtained as the range of \( \int \lambda(t) \, dP_\lambda \), and by \( P(F)^- \) its ultraweak closure in \( B(H) \).

**Theorem.** With the above assumptions and notation, if \( E \) is a closed subset of \( \Gamma \) then \( M^\alpha(E) = \bigcap \{ P(E + N)^- \} \), where the intersection is taken over all compact neighborhoods of the identity in \( \Gamma \).

**Proof.** Let \( F \) be a closed subset of \( \Gamma \). Let \( x \in M \cap P(F)^- \) and \( (x_\beta) \) be a net in \( P(F) \) which converges ultraweakly to \( x \). Let \( f \in L^1(G) \) have Fourier transform vanishing in a neighborhood of \( F \). By [1, Proposition 1.6] \( \pi_\alpha(f) \) is an ultraweakly continuous linear map on \( B(H) \). Thus by [1, Remark, §2] \( 0 = \pi_\alpha(f)(x_\beta) \to \pi_\alpha(f)(x) \), where we have identified the operator \( \pi_\alpha(f) \) defined by \( \hat{\alpha} \) on \( H_2 \) and its extension \( \pi_\alpha(f) \) to \( B(H) \). Thus \( \pi_\alpha(f)(x) = 0 \) for all such \( f \), so again by [1], \( x \in M^\alpha(F) \). Thus \( \bigcap \{ M \cap P(E + N)^- \} \subset \bigcap \{ M^\alpha(E + N) \} = M^\alpha(E) \) [1, Proposition 2.2].

Conversely, let \( x \in M^\alpha(E) \), so, in particular, \( x \in B(H)^\alpha(E) \). If \( F \) is a closed subset of \( \Gamma \) let \( R_\alpha^F(F) \) (resp. \( R^\alpha(F) \)) denote the closed (resp. ultraweakly closed) subspace of \( H_2 \) (resp. \( B(H) \)) generated by range \( \pi_\alpha(f) \) in \( H_2 \) (resp. in \( B(H) \)) for all \( f \in L^1(G) \) with \( \text{supp} \hat{f} \) compact and contained in \( F \). By [1, Proposition 2.2], \( B(H)^\alpha(E) = \bigcap \{ R_\alpha^F(E + N) \} \), where the intersection is taken over all compact neighborhoods \( N \) of the identity in \( \Gamma \). Let \( x \in R^\alpha(E + N) \) and assume there are \( f \in L^1(G) \) such that \( \text{supp} \hat{f} \) is compact and contained in \( E + N \), and \( y \in B(H) \), such that \( x = \pi_\alpha(f)(y) \). Since \( H_2 \) is ultraweakly dense in \( B(H) \), there is a net \( (y_\beta) \) in \( H_2 \) which converges ultraweakly to \( y \). Since \( \pi_\alpha(f) \) is ultraweakly continuous,

\[ x = \pi_\alpha(f)(y) = \lim_{\beta} \pi_\alpha(f)(y_\beta) \in R_\alpha^F(E + N) \]

But from the theory of spectral subspaces applied to unitary representations [1], \( R_\alpha^F(F) = P(F) \) for all closed sets \( F \subset \Gamma \). Thus \( x \in P(E + N)^- \) for all \( N \), and since such \( x \) are dense in \( R^\alpha(E + N) \), the proof is complete.

**Remark 1.** We cannot sharpen the theorem to a statement like \( "M^\alpha(E) = M \cap P(E)^-" \). Indeed, let \( H \) be a separable Hilbert space and \( u \) a unitary operator on \( H \) such that the von Neumann algebra \( A \) generated by \( u \) is a maximal abelian subalgebra of \( B(H) \) without minimal projections. Let \( \alpha \) be...
the representation of the integers defined by \( \alpha_n = \text{Ad}(u^n) \in \text{Aut} \ B(H) \). Then there is no nonzero \( x \in H_2 \) such that \( \alpha_n(x) = x \) for all \( n \), so \( P(\{1\}) = \{0\} \), while \( B(H)^a(\{1\}) = A \).

Remark 2. The main idea in the proof of the theorem was to consider the restriction \( \tilde{\phi} \) of a map \( \varphi \in B(B(H)) \), the bounded linear maps of \( B(H) \) into itself, to \( H_2 \). If \( \tilde{\phi} \) is a bounded normal linear operator on \( H_2 \) we can do spectral theory for \( \tilde{\phi} \) in \( B(H_2) \). It is tempting to generalize the above theorem and try to "lift" spectral theory for \( \tilde{\phi} \) in \( B(H_2) \) to that of \( \varphi \) in \( B(B(H)) \). This, however, seems to be quite hopeless except in special cases. Indeed, while the norm \( \|\varphi\| \) of \( \varphi \) in \( B(B(H)) \) is never smaller than the norm \( \|\tilde{\varphi}\| \) of \( \tilde{\varphi} \) in \( B(H_2) \), there is no finite constant \( k > 0 \) such that \( \|\varphi\| < k\|\tilde{\varphi}\| \) for all such \( \varphi \).

References


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