REPRESENTABLE MONOIDS

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Abstract. A representable monoid is one with enough representative functions to separate points. It is shown that the monoid algebra of a representable monoid is a proper algebra. In particular, the group algebra of a residually-finite group is a proper algebra. It is also shown that the free product of two representable monoids is again representable.

1. Introduction. Let $S$ be a monoid, $K$ a field. For $s$ in $S$, let $r_s$ and $l_s$ denote right and left multiplication by $s$ in $S$. A function $f: S \to K$ is called representative if the right translates $s \cdot f = f_{r_s}$ of $f$ for $s$ in $S$ span a finite-dimensional $K$-vector space. Let $R(S)$ denote the set of representative functions on $S$. $R(S)$ is a bialgebra (Hopf algebra, possibly without an antipode). $R$ is a contravariant functor from monoids to bialgebras. We called $S$ representable in [T I] if $R(S)$ separates points in $S$. In this note, we study the class of representable monoids. See [H] for a discussion of representative functions. There $S$ is a group, but the basic ideas carry over for $S$ a monoid.

There is a close relationship between representable monoids and proper algebras. Recall that an algebra $A$ is called proper if there are enough linear functions representative with respect to the multiplicative monoid structure of $A$ to separate points of $A$. See [S], [T I] and [T II] for alternate characterizations of proper algebras. We show here that $S$ is representable if and only if the monoid algebra $K[S]$ is proper. We use this result to show that the coproduct of two representable monoids is representable.

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2. The class of representable monoids. $K[S]$ will denote the monoid algebra of $S$ over $K$.

Lemma 1. $S$ is representable if and only if for any $s \neq t$ in $S$, there is a representation $\rho$ of $S$ on a finite-dimensional $K$-vector space for which $\rho(s) \neq \rho(t)$.

Proof. Let $S$ be representable, and $s \neq t$ in $S$. Choose $f$ in $R(S)$ with $f(s) \neq f(t)$. Then $K[S] \cdot f$, the span of the right translates of $f$ by elements of
$S$ is a (left)$K[S]$-module which is finite-dimensional over $K$. $(s \cdot f)(1) = f(s)$ and $(t \cdot f)(1) = f(t)$, so $s \cdot f \neq t \cdot f$ and $\rho(s) \neq \rho(t)$, where $\rho$ is the representation of $S$ on $K[S] \cdot f$. Conversely, for $s \neq t$ in $S$, let $\rho$ be a representation of $S$ on a finite-dimensional space $V$ so that $\rho(s) \neq \rho(t)$. If $p_y$ is a coordinate function on $\text{Hom}_K(V, V)$ separating $\rho(s)$ and $\rho(t)$, then $p_y \rho$ is a representative function on $S$ separating $s$ and $t$. This follows from the formula for matrix multiplication. For if $a, b$ are in $S$, then

$$(p_y \rho a)(b) = p_y(ba) = p_y(\rho(b) \rho(a)) = \sum_k (p_{ik} \rho(b))(p_{ik} \rho(a)).$$

Thus $p_y \rho a = \sum_k (p_{ik} \rho(a))(p_{ik} \rho)$ lies in the span of the $p_{ik} \rho$.

We included Lemma 1 for completeness. The idea is that $R(S)$ consists of all coordinate functions arising from finite-dimensional representations of $S$. See [H, pp. 14–15], for a discussion of this.

**Proposition 2.** A submonoid of a representable monoid is representable.

**Proof.** This is clear by restriction of finite-dimensional representations.

**Proposition 3.** Let $\{S_i | i \in I\}$ be a collection of monoids. Then $S = \prod_{i} S_i$ is representable if and only if each $S_i$ is representable.

**Proof.** If $S$ is representable, then each $S_i$ is representable by Proposition 2. Conversely let each $S_i$ be representable. For $s \neq t$ in $S$, let $p_j$ be a projection of $S$ onto $S_j$ for which $p_j(s) \neq p_j(t)$. Let $f \in R(S_j)$, $f(p_j(s)) \neq f(p_j(t))$. Then $fp_j$ is in $R(S)$ and separates $s$ and $t$.

**Corollary 4.** A direct sum (weak direct product) of monoids $\{S_i | i \in I\}$ is representable if and only if each $S_i$ is representable.

**Proof.** This follows from Propositions 2 and 3.

**Lemma 5.** Let $A$ be a $K$-algebra which is an integral domain. Let $\{a_i | 1 \leq i \leq n\}$ be distinct elements of $A$. Then the elements $\{(1, a_i, a_i^2, \ldots, a_i^{n-1}) | 1 \leq i \leq n\}$ of $A^n$ are linearly independent over $K$.

**Proof.** By the Vandermonde determinant, $\{(1, a_i, a_i^2, \ldots, a_i^{n-1}) | 1 \leq i \leq n\}$ are linearly independent over $A$, hence also over $K$.

**Lemma 6.** Let $V$ be a vector space over $K$. For $i > 0$, let $V^{(i)}$ denote $V \otimes V \otimes \cdots \otimes V$ ($i$ factors). If $v_1, \ldots, v_n$ are distinct elements of $V$, then the elements $\{(1, v_i, v_i \otimes v_i, \ldots, v_i \otimes \cdots \otimes v_i) | 1 \leq i \leq n\}$ of $K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{n-1}$ are linearly independent over $K$.

**Proof.** Let $S^{(i)}$ denote the $i$th symmetric tensor product of $V$. Then the elements $\{(1, v_i, v_i^2, \ldots, v_i^{n-1}) | 1 \leq i \leq n\}$ of the symmetric algebra $S(V)$ are linearly independent over $K$ by Lemma 5. Since $S(V)$ is a homomorphic image of the tensor algebra $T(V)$, the elements $\{(1, v_i, v_i \otimes v_i, \ldots, v_i \otimes \cdots \otimes v_i) | 1 \leq i \leq n\}$ of $T(V)$ are also linearly independent over $K$.  

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Lemma 7. Let $W$ be a finite-dimensional vector space over $K$. Let $\{T_i\,|\,1 < i < n\}$ be distinct elements of $V = \text{Hom}_K(W, W)$. For $1 < i < n$, let $\tilde{T}_i$ denote the diagonal extension of $T_i$ to $\text{Hom}_K(X, X)$, where $X = K \oplus W \oplus W^{(2)} \oplus \cdots \oplus W^{(n-1)} \subset T(W)$. Then $\{\tilde{T}_i\,|\,1 < i < n\}$ are linearly independent over $K$.

Proof. By Lemma 6, $\{(1, T_1, T_1 \otimes T_2, \ldots, T_1 \otimes \cdots \otimes T_n)\,|\,1 < i < n\}$ of $K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$ are linearly independent over $K$. Identifying $(\text{hom}_K(W, W))^{(j)}$ with $\text{Hom}_K(W^{(j)}, W^{(j)})$ for $0 < j < n - 1$, the element $(1, T_i, T_i \otimes T_j, \ldots, T_i \otimes \cdots \otimes T_n)$ corresponds to $\tilde{T}_i$. Thus $\{\tilde{T}_i\,|\,1 < i < n\}$ are linearly independent over $K$.

Theorem 8. A monoid $S$ is representable if and only if the monoid algebra $K[S]$ is proper.

Proof. If $K[S]$ is proper, then $S$ is representable by Proposition 3.6 of [TI]. Conversely, let $S$ be representable. Let $\alpha_1s_1 + \cdots + \alpha_ns_n$ be a nonzero element of $K[S]$, where $s_1, \ldots, s_n$ are distinct in $S$, and $\alpha_1, \ldots, \alpha_n$ are nonzero elements of $K$. For $i \neq j$, $1 < i, j < n$, let $\rho_{ij}$ be a representation of $S$ on a finite-dimensional vector space $V_{ij}$ for which $\rho_{ij}(s_i) \neq \rho_{ij}(s_j)$. Set $V = \Sigma_{ij=1}^n \oplus V_{ij}$, and $\rho = \Sigma_{i,j=1}^n \oplus \rho_{ij}$. Then $V$ is finite-dimensional over $K$ and $\rho(s_i) \neq \rho(s_j)$ for $i \neq j$, $1 < i, j < n$. Let $\tilde{\rho}$ be the diagonal extension of $\rho$ to a representation of $S$ and $K[S]$ on the space $W = K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$. Then $\{\tilde{\rho}(s_i)\,|\,1 < i < n\}$ are linearly independent by Lemma 7. Hence $\tilde{\rho}(\Sigma_{i=1}^n \alpha_is_i) = \Sigma_{i=1}^n \alpha_i\tilde{\rho}(s_i) \neq 0$ in $\text{Hom}_K(W, W)$. Hence the kernel of $\tilde{\rho}$ is a cofinite ideal of $K[S]$ not containing $\Sigma_{i=1}^n \alpha_is_i$. Hence $K[S]$ is proper by Lemma 6.1.0 of [S].

Corollary 9. If $S$ is a residually finite monoid, then $K[S]$ is a proper algebra.

Proof. We show $S$ is representable, so that the result follows from Theorem 8. Let $s \neq t$ in $S$. Let $\rho : S \to H$ be a homomorphism of $S$ to a finite monoid $H$ so that $\rho(s) \neq \rho(t)$. $\rho$ induces a representation $\tilde{\rho}$ of $S$ on $K[H]$ by left translation. $\tilde{\rho}(s)(1) = \rho(s) \neq \rho(t) = \tilde{\rho}(t)(1)$, so $\tilde{\rho}(s) \neq \tilde{\rho}(t)$.

If $G$ is a residually finite group, then it is residually finite as a monoid, and so $K[G]$ is a proper algebra. This was noted by A. Rosenberg in Theorem 2 of [ROS].

Theorem 10. Let $S$ and $T$ be representable monoids. Then the free product $S \ast T$ is a representable monoid.

Proof. $K[S]$ and $K[T]$ are proper by Theorem 8. Hence $K[S] \ast K[T]$ (the free product of algebras) is proper by Proposition 3.4.2 of [TIII]. But $K[S] \ast K[T] = K[S \ast T]$. Hence $S \ast T$ is representable by Theorem 8.

We conclude by noting that the class of representable monoids is not closed under homomorphic images. This was pointed out in Proposition 2.6 of [TI] if $K$ is a finite field, but we give examples here for any field $K$. First
note that any free monoid is representable by Proposition 2.5 of [T I]. Hence it suffices to exhibit a nonrepresentable monoid. Let $G$ be a finitely-generated infinite simple group. For examples of such, see §5.1 of [ROB]. We claim that $R(G) = K$, the constant functions on $G$, so that $G$ is not representable. Let $\rho$ be a (monoid) representation of $G$ on a finite-dimensional vector space of dimension $n$. $\rho(G)$ is a linear group of $n$ by $n$ matrices over $K$. Since $G$ is a simple group, $\rho$ is either one-to-one or trivial. If $\rho$ is one-to-one, then $\rho(G)$ is a finitely-generated infinite linear group. By Malcev's theorem (see Corollary 4.4(ii) of [W]), $\rho(G)$ is not simple. Hence $\rho$ is trivial, i.e. a one-dimensional representation. It follows that $R(G) = K$.

References


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