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REPRESENTABLE MONOIDS

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ABSTRACT. A representable monoid is one with enough representative functions to separate points. It is shown that the monoid algebra of a representable monoid is a proper algebra. In particular, the group algebra of a residually-finite group is a proper algebra. It is also shown that the free product of two representable monoids is again representable.

1. Introduction. Let S be a monoid, K a field. For s in S, let r_s and l_s denote right and left multiplication by s in S. A function f: $S \to K$ is called representative if the right translates $s \cdot f = fr_s$ of f for s in S span a finite-dimensional K-vector space. Let $R(S)$ denote the set of representative functions on S. $R(S)$ is a bialgebra (Hopf algebra, possibly without an antipode). R is a contravariant functor from monoids to bialgebras. We called S representable in [T I] if $R(S)$ separates points in S. In this note, we study the class of representable monoids. See [H] for a discussion of representative functions. There S is a group, but the basic ideas carry over for S a monoid.

There is a close relationship between representable monoids and proper algebras. Recall that an algebra A is called proper if there are enough linear functions representative with respect to the multiplicative monoid structure of A to separate points of A. See [S], [T I] and [T II] for alternate characterizations of proper algebras. We show here that S is representable if and only if the monoid algebra $K[S]$ is proper. We use this result to show that the coproduct of two representable monoids is representable.

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2. The class of representable monoids. $K[S]$ will denote the monoid algebra of S over K.

LEMMA 1. *S is representable if and only if for any $s \neq t$ in S, there is a representation ρ of S on a finite-dimensional K-vector space for which $\rho(s) \neq \rho(t)$.*

PROOF. Let S be representable, and $s \neq t$ in S. Choose f in $R(S)$ with $f(s) \neq f(t)$. Then $K[S] \cdot f$, the span of the right translates of f by elements of

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S is a (left)$K[S]$-module which is finite-dimensional over K. $(s\cdot f)(1) = f(s)$ and $(t\cdot f)(1) = f(t)$, so $s\cdot f \neq t\cdot f$ and $\rho(s) \neq \rho(t)$, where ρ is the representation of S on $K[S]\cdot f$. Conversely, for $s \neq t$ in S, let ρ be a representation of S on a finite-dimensional space V so that $\rho(s) \neq \rho(t)$. If p_{ij} is a coordinate function on $\mathrm{Hom}_K(V, V)$ separating $\rho(s)$ and $\rho(t)$, then $p_{ij}\rho$ is a representative function on S separating s and t. This follows from the formula for matrix multiplication. For if a, b are in S, then

$$\begin{aligned}(p_{ij}\rho r_a)(b) &= p_{ij}\rho(ba) = p_{ij}(\rho(b)\rho(a))\\ &= \sum_k (p_{ik}\rho(b))(p_{kj}\rho(a)).\end{aligned}$$

Thus $p_{ij}\rho r_a = \sum_k (p_{kj}\rho(a))(p_{ik}\rho)$ lies in the span of the $p_{ik}\rho$.

We included Lemma 1 for completeness. The idea is that $R(S)$ consists of all coordinate functions arising from finite-dimensional representations of S. See [**H**, pp. 14–15], for a discussion of this.

PROPOSITION 2. *A submonoid of a representable monoid is representable.*

PROOF. This is clear by restriction of finite-dimensional representations.

PROPOSITION 3. *Let $\{S_i | i \text{ in } I\}$ be a collection of monoids. Then $S = \prod_I S_i$ is representable if and only if each S_i is representable.*

PROOF. If S is representable, then each S_i is representable by Proposition 2. Conversely let each S_i be representable. For $s \neq t$ in S, let p_j be a projection of S onto S_j for which $p_j(s) \neq p_j(t)$. Let $f \in R(S_j)$, $f(p_j(s)) \neq f(p_j(t))$. Then fp_j is in $R(S)$ and separates s and t.

COROLLARY 4. *A direct sum (weak direct product) of monoids $\{S_i | i \text{ in } I\}$ is representable if and only if each S_i is representable.*

PROOF. This follows from Propositions 2 and 3.

LEMMA 5. *Let A be a K-algebra which is an integral domain. Let $\{a_i | 1 \leqslant i \leqslant n\}$ be distinct elements of A. Then the elements $\{(1, a_i, a_i^2, \dots, a_i^{n-1}) | 1 \leqslant i \leqslant n\}$ of A^n are linearly independent over K.*

PROOF. By the Vandermonde determinant, $\{(1, a_i, a_i^2, \dots, a_i^{n-1}) | 1 \leqslant i \leqslant n\}$ are linearly independent over A, hence also over K.

LEMMA 6. *Let V be a vector space over K. For $i \geqslant 0$, let $V^{(i)}$ denote $V \otimes V \otimes \cdots \otimes V$ (i factors). If $v_1, \dots, v_n$ are distinct elements of V, then the elements $\{(1, v_i, v_i \otimes v_i, \dots, v_i \otimes \cdots \otimes v_i) | 1 \leqslant i \leqslant n\}$ of $K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{n-1}$ are linearly independent over K.*

PROOF. Let $S^{(i)}$ denote the ith symmetric tensor product of V. Then the elements $\{(1, v_i, v_i^2, \dots, v_i^{n-1}) | 1 \leqslant i \leqslant n\}$ of the symmetric algebra $S(V)$ are linearly independent over K by Lemma 5. Since $S(V)$ is a homomorphic image of the tensor algebra $T(V)$, the elements $\{(1, v_i, v_i \otimes v_i, \dots, v_i \otimes \cdots \otimes v_i) | 1 \leqslant i \leqslant n\}$ of $T(V)$ are also linearly independent over K.

LEMMA 7. *Let W be a finite-dimensional vector space over K. Let $\{T_i|1 \leqslant i \leqslant n\}$ be distinct elements of $V = \mathrm{Hom}_K(W, W)$. For $1 \leqslant i \leqslant n$, let $\tilde{T}_i$ denote the diagonal extension of T_i to $\mathrm{Hom}_K(X, X)$, where $X = K \oplus W \oplus W^{(2)} \oplus \cdots \oplus W^{(n-1)} \subset T(W)$. Then $\{\tilde{T}_i|1 \leqslant i \leqslant n\}$ are linearly independent over K.*

PROOF. By Lemma 6, $\{(1, T_i, T_i \otimes T_i, \ldots, T_i \otimes \cdots \otimes T_i)|1 \leqslant i \leqslant n\}$ of $K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$ are linearly independent over K. Identifying $(\mathrm{hom}_K(W, W))^{(j)}$ with $\mathrm{Hom}_K(W^{(j)}, W^{(j)})$ for $0 \leqslant j \leqslant n-1$, the element $(1, T_i, T_i \otimes T_i, \ldots, T_i \otimes \cdots \otimes T_i)$ corresponds to $\tilde{T}_i$. Thus $\{\tilde{T}_i|1 \leqslant i \leqslant n\}$ are linearly independent over K.

THEOREM 8. *A monoid S is representable if and only if the monoid algebra $K[S]$ is proper.*

PROOF. If $K[S]$ is proper, then S is representable by Proposition 3.6 of [T I]. Conversely, let S be representable. Let $\alpha_1 s_1 + \cdots + \alpha_n s_n$ be a nonzero element of $K[S]$, where $s_1, \ldots, s_n$ are distinct in S, and $\alpha_1, \ldots, \alpha_n$ are nonzero elements of K. For $i \neq j$, $1 \leqslant i, j \leqslant n$, let ρ_{ij} be a representation of S on a finite-dimensional vector space V_{ij} for which $\rho_{ij}(s_i) \neq \rho_{ij}(s_j)$. Set $V = \sum_{i,j=1}^{n} \oplus V_{ij}$, and $\rho = \sum_{i,j=1}^{n} \oplus \rho_{ij}$. Then V is finite-dimensional over K and $\rho(s_i) \neq \rho(s_j)$ for $i \neq j$, $1 \leqslant i, j \leqslant n$. Let $\tilde{\rho}$ be the diagonal extension of ρ to a representation of S and $K[S]$ on the space $W = K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)}$. Then $\{\tilde{\rho}(s_i)|1 \leqslant i \leqslant n\}$ are linearly independent by Lemma 7. Hence $\tilde{\rho}(\sum_{i=1}^{n} \alpha_i s_i) = \sum_{i=1}^{n} \alpha_i \tilde{\rho}(s_i) \neq 0$ in $\mathrm{Hom}_K(W, W)$. Hence the kernel of $\tilde{\rho}$ is a cofinite ideal of $K[S]$ not containing $\sum_{i=1}^{n} \alpha_i s_i$. Hence $K[S]$ is proper by Lemma 6.1.0 of $[S]$.

COROLLARY 9. *If S is a residually finite monoid, then $K[S]$ is a proper algebra.*

PROOF. We show S is representable, so that the result follows from Theorem 8. Let $s \neq t$ in S. Let $\rho : S \to H$ be a homomorphism of S to a finite monoid H so that $\rho(s) \neq \rho(t)$. ρ induces a representation $\bar{\rho}$ of S on $K[H]$ by left translation. $\bar{\rho}(s)(1) = \rho(s) \neq \rho(t) = \bar{\rho}(t)(1)$, so $\bar{\rho}(s) \neq \bar{\rho}(t)$.

If G is a residually finite group, then it is residually finite as a monoid, and so $K[G]$ is a proper algebra. This was noted by A. Rosenberg in Theorem 2 of [**ROS**].

THEOREM 10. *Let S and T be representable monoids. Then the free product $S * T$ is a representable monoid.*

PROOF. $K[S]$ and $K[T]$ are proper by Theorem 8. Hence $K[S] * K[T]$ (the free product of algebras) is proper by Proposition 3.4.2 of [**T II**]. But $K[S] * K[T] = K[S * T]$. Hence $S * T$ is representable by Theorem 8.

We conclude by noting that the class of representable monoids is not closed under homomorphic images. This was pointed out in Proposition 2.6 of [**T I**] if K is a finite field, but we give examples here for any field K. First note that any free monoid is representable by Proposition 2.5 of **[T I]**. Hence it suffices to exhibit a nonrepresentable monoid. Let G be a finitely-generated infinite simple group. For examples of such, see §5.1 of **[ROB]**. We claim that $R(G) = K$, the constant functions on G, so that G is not representable. Let ρ be a (monoid) representation of G on a finite-dimensional vector space of dimension n. $\rho(G)$ is a linear group of n by n matrices over K. Since G is a simple group, ρ is either one-to-one or trivial. If ρ is one-to-one, then $\rho(G)$ is a finitely-generated infinite linear group. By Malcev's theorem (see Corollary 4.4(ii) of **[W]**), $\rho(G)$ is not simple. Hence ρ is trivial, i.e. a one-dimensional representation. It follows that $R(G) = K$.

References

[H] G. Hochschild, *Introduction to affine algebraic groups*, Holden-Day, San Francisco, Calif., 1971.

[ROB] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups*, Part 1, Ergebnisse der Math., Band 62, Springer-Verlag, Berlin and New York, 1972.

[ROS] A. Rosenberg, *On the application of coalgebras to group algebras*, Proc. Amer. Math. Soc. **52** (1975), 109–112.

[S] M. Sweedler, *Hopf algebras*, Benjamin, New York, 1969.

[T I] E. J. Taft, *Reflexivity of algebras and coalgebras*, Amer. J. Math. **94** (1972), 1111–1130.

[T II] ______, *Reflexivity of algebras and coalgebras*.II, Comm. Algebra **5** (1977), 1549–1560.

[W] B. A. F. Wehrfritz, *Infinite linear groups*, Ergebnisse der Math., Band 76, Springer-Verlag, Berlin and New York, 1973.

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