A GENERALIZATION OF A THEOREM OF TATCHELL

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ABSTRACT. Necessary and sufficient conditions for $\sum a_n\epsilon_n$ to be summable $|A, \lambda_n|$, whenever $\sum a_n$ is convergent, have been obtained. The sufficiency part of this result has also been improved.

1. Let $\sum_{n=0}^{\infty}a_n$ be a given infinite series and let $\{\lambda_n\}$ be a sequence of nonnegative numbers increasing to infinity. For any $k > 0$, we write

$$A^k(t) = \sum_{\lambda_n < t} (t - \lambda_n)^k a_n.$$ 

We say that $\sum_{n=0}^{\infty}a_n$ is summable $(R, \lambda_n, k)$ to $S$ if $t^{-k}A^k(t) \to S$ as $t \to \infty$. If, in addition, $t^{-k}A^k(t)$ is of bounded variation in $(0, \infty)$, we say that the series $\sum_{n=0}^{\infty}a_n$ is summable $|R, \lambda_n, k|$ to $S$. The discontinuous Riesz means $(R^*, \lambda_n, k)$ are obtained by restricting $t$ to the sequence $\{\lambda_n\}$. If $\sum a_n e^{-\lambda_n x}$ is convergent for all positive $x$ and $f(x) = \sum a_n e^{-\lambda_n x} \to S$ when $x \to 0$, then we say that the series $\sum_{n=0}^{\infty}a_n$ is summable $(A, \lambda_n)$ to sum $S$, and write $\sum_{n=0}^{\infty}a_n = S(A, \lambda_n)$. When $\lambda_n = n$, the $(A, \lambda_n)$ method is the Abel method. The series $\sum_{n=0}^{\infty}a_n$ is said to be absolutely summable $(A, \lambda_n)$ or summable $|A, \lambda_n|$, if the series $\sum_{n=0}^{\infty}a_n e^{-\lambda_n x}$ is convergent for all positive values of $x$ and the sum function $f(x) = \sum_{n=0}^{\infty}a_n e^{-\lambda_n x}$ is of bounded variation in $[0, \infty)$. We also define $\Delta \epsilon_n = \epsilon_n - \epsilon_{n+1}$.

2. Necessary and sufficient conditions on $\{\epsilon_n\}$ in order that $\sum a_n\epsilon_n$ should be absolutely Abel summable whenever $\sum a_n$ converges were obtained by Tatchell [3] in 1954 in the form of the following

**Theorem.** Necessary and sufficient conditions for $\sum a_n\epsilon_n$ to be summable $|A|$, whenever $\sum a_n$ is convergent are

$$\sum |\Delta \epsilon_n| < \infty, \quad (2.1)$$

and

$$\sum |\epsilon_n|/n < \infty. \quad (2.2)$$

The object of this paper is to generalize the above theorem by replacing absolute Abel summability $|A|$ by summability $|A, \lambda_n|$ for any sequence $\{\lambda_n\}$ satisfying weaker conditions. In what follows we prove the following theorem.

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Theorem 1. Necessary and sufficient conditions for $\sum a_n e_n$ to be summable $|A, \lambda_n|$, whenever $\sum a_n$ is convergent, are

$$\sum |\Delta e_n| < \infty, \quad (2.3)$$
$$\sum |e_n|(1 - \mu_n) < \infty, \quad (2.4)$$

where

$$\mu_n = \frac{\lambda_n}{\lambda_{n+1}} \quad \text{and} \quad \lambda_{n+1} = O(\lambda_n).$$

Taking $\lambda_n = n$ in our theorem, the above-mentioned theorem of Tatchell follows as a special case.

3. To prove the theorem we need the following lemmas:

Lemma 1. If $H(p) = \alpha_p$ is a transformation from a Banach space $B$ to the space $L$, and if $h_x(p) = \alpha_x(p)$ is a continuous linear functional on $B$ whenever $x > 0$, then $H(p)$ is a bounded linear operator.

$L$ in this context is really $\hat{L}$ as defined in Dunford and Schwartz, Linear operators. I, p. 119.

The proof of this lemma may be found in [4].

Lemma 2. If a sequence $\{e_n\}$ has the property that the function

$$\sum_{n=0}^{\infty} S_n e_n \left\{ \frac{d}{dx} e^{-\lambda_n x} \right\}$$

is defined and has a finite Lebesgue integral on $[0, \infty)$, whenever $\{S_n\}$ is a convergent sequence, then there is a number $\tilde{H}$ such that

$$\int_0^{\infty} \sum_{n=0}^{\infty} S_n e_n \left\{ \frac{d}{dx} e^{-\lambda_n x} \right\} dx < \tilde{H} \bar{d} |S_n|$$

for every convergent sequence $\{S_n\}$.

Proof. If

$$h_x(S) = \sum_{n=0}^{\infty} S_n e_n \left\{ \frac{d}{dx} e^{-\lambda_n x} \right\}$$

is defined whenever $S = \{S_n\}$ is a convergent sequence, then $h_x$ is a linear functional on the Banach space $C$. Therefore, by hypothesis, $h_x$ is a linear functional on $C$ whenever $0 < x < \infty$.

Also by hypothesis

$$\sum_{n=0}^{\infty} S_n e_n \left\{ \frac{d}{dx} e^{-\lambda_n x} \right\}$$

is in the space $L$ whenever $S$ is in $C$. Hence the lemma follows from Lemma 1.

Lemma 3. If a sequence $\{p_n\}$ of elements in a Banach space $B$ has the property that there is a number $H$ such that for every nonnegative integer $K$ and
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every set of real numbers \( \theta_0, \theta_1, \theta_2, \ldots, \theta_k \),

\[
\left\| \sum_{n=0}^{k} e^{i\theta_n} p_n \right\| < H,
\]

then \( \sum_{n=0}^{\infty} |f(p_n)| < \infty \), for every linear functional \( f \) on \( B \).

**Proof.** Let \( f \) be a linear functional on \( B \), and let \( \theta_n = - \arg f(p_n) \).

\[
\sum_{n=0}^{k} |f(p_n)| = \sum_{n=0}^{k} e^{i\theta_n} f(p_n) < H\|f\|,
\]

whence the result follows.

**Lemma 4.** The necessary and sufficient conditions that \( \gamma(a) = \sum_{k=0}^{\infty} g_k(a) c_k \) should tend to a finite limit as \( a \to \infty \) whenever \( \sum c_k \) is convergent are:

(i) \( \sum_{k=0}^{\infty} \|g_k(a) - g_{k+1}(a)\| < M \) independently of \( a > a' \);

(ii) \( \lim_{a \to \infty} g_k(a) = \beta_k \) for every fixed \( k \).

The proof of this lemma may be found in [5].

**Lemma 5.** A necessary and sufficient condition for \( \sum a_n e_n \) to be summable \((A, \lambda_n)\) whenever \( \sum a_n \) is convergent is that \( \sum |\Delta e_n| < \infty \).

It may be remarked that if we take \( \lambda_n = n \) we get the following result of Bosanquet [Proc. London Math. Soc. (2) 50 (1948), Lemma 9].

**Lemma A.** A necessary and sufficient condition for \( \sum a_n e_n \) to be summable \((A)\) whenever \( \sum a_n \) is convergent is that \( \sum |\Delta e_n| < \infty \).

**Proof of Lemma 5.** Since \( \sum a_n e_n \) is summable \((A, \lambda_n)\) we have

\[
\sum_{n=1}^{\infty} a_n e_n e^{-\lambda_n/t} \quad \text{is convergent for } t > 0
\]

and

\[
\lim_{t \to \infty} \sum_{n=1}^{\infty} a_n e_n e^{-\lambda_n/t} \quad \text{exists.}
\]

Applying Lemma 4 we have

\[
\sum_{n=1}^{\infty} \left| \Delta(e_n e^{-\lambda_n/t}) \right| < M \quad \text{independently of } t > t_0.
\]

This implies that \( \sum_{n=1}^{\infty} |\Delta e_n| < \infty \).

The proof of the sufficiency part of the lemma is immediate for \( \sum a_n \) is convergent and \( \{e_n\} \) is a sequence of bounded variation.

**Lemma 6.** We have

\[
\int_0^{\infty} |d\Delta e^{-\lambda_n x}| = 2(1 - \mu_n)(\mu_n)^{\mu_n/(1-\mu_n)}.
\]

The proof is simple.
Lemma 7. If \( \mu_n = \lambda_n / \lambda_{n+1} \), then the following conditions are equivalent.

(i) \( \sum |e_n| (1 - \mu_n) < \infty \),

(ii) \( \sum |e_n| (1 - \mu_n) \mu_n^{1/(1 - \mu)} < \infty \).

Proof. Since \( 0 < \mu_n < 1 \), so \( e^{-1} < \mu_n^{1/(1 - \mu)} < 1 \). Hence (i) and (ii) are equivalent.

Lemma 8. Let \( M, a \) be constants with \( M > 1, a > 0 \) and

(a) \( \log M < 2\pi \).

Then for \( 1/M < u < 1 \) we have \( |1 - u^{-a}|/(1 - u) > C \), where \( C > 0 \) is a constant (depending on \( a \) and \( M \)).

Proof. Given (i), the function \( (1 - u^{-a})(1 - u)^{-1} \) is continuous and nonzero on \( 1/M < u < 1 \), and has a nonzero limit as \( u \to 1 \). This proves the lemma.

Lemma 9. For \( k = 1 \), \( |R, \lambda_n, k| \sim |R^*, \lambda_n, k| \).

Proof. Let

\[ \gamma(t) = t^{-k} \sum_{\lambda_n < t} (t - \lambda_n)^k a_n. \]

\( |R, \lambda_n, k| \) means that

\[ \int_0^\infty \left| d\gamma(t) \right| < \infty. \quad (i) \]

\( |R^*, \lambda_n, k| \) means

\[ \sum_{n=1}^\infty |\gamma(\lambda_n) - \gamma(\lambda_{n+1})| < \infty. \quad (ii) \]

Now if \( k = 1 \), then for \( \lambda_n < t < \lambda_{n+1} \),

\[ \gamma(t) = \frac{1}{t} \sum_{r=0}^n (t - \lambda_r) a_r = A - B / t, \]

where \( A = \sum_{r=0}^n a_r \), \( B = \sum_{r=0}^n \lambda_r a_r \). Thus, for a given \( n \), \( A, B \) are constants (though their values will vary with \( n \)).

Hence, for a given \( n \), \( \gamma(t) \) is monotonic in each interval \([\lambda_n, \lambda_{n+1}]\) (increasing if \( B > 0 \) and decreasing if \( B < 0 \)). Hence

\[ \int_{\lambda_n}^{\lambda_{n+1}} |d\gamma(t)| = |\gamma(\lambda_{n+1}) - \gamma(\lambda_n)| \]

so that (i) and (ii) are equivalent.

Note. It is known [6] that \( |R, \lambda_n, k| \sim |R^*, \lambda_n, k| \) for \( 0 < k < 1 \).

Lemma 10. For any \( k > 0 \), \( |R, \lambda_n, k| \Rightarrow |A, \lambda_n| \).

The proof of this lemma may be found in [7].

4. Proof of the theorem.

Sufficiency. Since \( \Sigma a_n \) is convergent, by virtue of (2.3) and Abel's test we see that \( \Sigma a_n \epsilon_n e^{-\lambda_n x} \) is convergent for all positive \( x \), so that writing
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\[ \alpha(x) = \sum_{n=0}^{\infty} a_n e_n e^{-\lambda_n x}, \]

we have

\[ \alpha(x) = \sum_{n=0}^{\infty} S_n \Delta(e_n e^{-\lambda_n x}), \quad S_n = a_0 + a_1 + \cdots + a_n \]

\[ = \sum_{n=0}^{\infty} S_n e^{-\lambda_{n+1} x} \Delta e_n + \sum_{n=0}^{\infty} S_n e_n \Delta e^{-\lambda_n x} \]

\[ = \alpha_1(x) + \alpha_2(x) \quad (\text{say}). \quad (4.1) \]

In order to prove that \( \Sigma a_n e_n \) is summable \( |A, \lambda_n| \) it is sufficient to show that \( \alpha_1(x) \) and \( \alpha_2(x) \) are functions of bounded variation on \([0, \infty)\), that is to say,

\[ \int_0^\infty |d\alpha_1(x)| < \infty, \quad (4.2) \]

and

\[ \int_0^\infty |d\alpha_2(x)| < \infty. \quad (4.3) \]

Now

\[ \int_0^\infty |d\alpha_1(x)| = \int_0^\infty \left| d \sum_{n=0}^{\infty} S_n e^{-\lambda_{n+1} x} \Delta e_n \right| \]

\[ < \sum_{n=0}^{\infty} |S_n| \left| \Delta e_n \right| \int_0^\infty |d e^{-\lambda_{n+1} x}| \]

\[ < \frac{bd}{\lambda} |S_n| \sum_{n=0}^{\infty} |\Delta e_n| < \infty. \]

Also by virtue of Lemmas 6 and 7 and condition (2.4) we observe that

\[ \int_0^\infty |d\alpha_2(x)| = \int_0^\infty \left| d \sum_{n=0}^{\infty} S_n e_n \Delta e^{-\lambda_n x} \right| \]

\[ < \sum_{n=0}^{\infty} |S_n| \left| e_n \right| \int_0^\infty |d \Delta e^{-\lambda_n x}| \]

\[ < 2 \frac{bd}{\lambda} |S_n| \sum_{n=0}^{\infty} |e_n| (1 - \mu_n) \mu_n^{\lambda_n/(1-\mu_n)} \]

\[ = 2 \frac{bd}{\lambda} |S_n| \sum_{n=0}^{\infty} |e_n| (1 - \mu_n) < \infty. \]

This completes the sufficiency part of the theorem.

\textit{Necessity.} Since summability \( |A, \lambda_n| \) implies summability \( (A, \lambda_n) \), it follows from Lemma 5 that (2.3) is necessary. Also from (4.1), since (2.3) holds, we have, as before, \( \int_0^\infty |d\alpha_1(x)| < \infty \).

Since by hypothesis \( \alpha(x) \) is of bounded variation it follows that \( \alpha_2(x) \) is also of bounded variation. Therefore
\[
\int_0^\infty \left| \sum_{n=0}^{\infty} S_n e_n \left[ \frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| \, dx = \int_0^\infty \left| \frac{d}{dx} \sum_{n=0}^{\infty} S_n e_n \Delta e^{-\lambda_n x} \right| \, dx \\
= \int_0^\infty \left| \frac{d}{dx} \sum_{n=0}^{\infty} S_n e_n e^{-\lambda_n x} \right| < \infty \quad (4.4)
\]
for every convergent sequence \( \{S_n\} \). Applying Lemma 2 we have
\[
\int_0^\infty \left| \sum_{n=0}^{\infty} S_n e_n \left[ \frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| \, dx < H \overline{b(d|S_n|)}
\]
for every sequence \( \{S_n\} \). In particular, we have
\[
\int_0^\infty \left| \sum_{n=0}^{k} e^{i\theta_n} e_n \left[ \frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| \, dx < H \quad (4.5)
\]
for every nonnegative integer \( k \) and every set of real numbers \( \theta_0, \theta_1, \ldots, \theta_k \). Again the sequence \( \{e_n(d/dx)\Delta e^{-\lambda_n x}\} \in L \), and so
\[
\left\| \sum_{n=0}^{k} e^{i\theta_n} e_n \frac{d}{dx} \Delta e^{-\lambda_n x} \right\| < H,
\]
by virtue of inequality (4.5). Hence, by Lemma 3 we have
\[
\left| \sum_{n=0}^{\infty} f\left( e_n \frac{d}{dx} \Delta e^{-\lambda_n x} \right) \right| < \infty
\]
for every linear functional \( f \) on \( L \).

But, for a given bounded measurable complex function \( \phi(x) \),
\[
f_{\phi}(\psi) = \int_0^\infty \phi(x)\psi(x) \, dx
\]
is a linear functional on \( L \). Therefore
\[
\left| \sum_{n=0}^{\infty} \phi(x)e_n \frac{d}{dx} \Delta e^{-\lambda_n x} \right| < \infty.
\]
This implies that
\[
\sum_{n=0}^{\infty} \left| e_n \right| \left| \int_0^\infty \phi(x) \frac{d}{dx} \Delta e^{-\lambda_n x} \right| < \infty. \quad (4.6)
\]
Since \( \lambda_{n+1} = O(\lambda_n) \) so there exists some constant \( M \) such that, for all \( n > 1 \), \( \lambda_{n+1} < M \lambda_n \). Given this \( M \), choose \( a > 0 \) so that \( a \log M < 2\pi \). Then apply (4.6) with \( \phi(x) = x^a \). (This choice is suggested by the argument of [3].) Now
\[
\int_0^\infty \phi(x)d(\Delta e^{-\lambda_n x}) = \Gamma(1 + ia)(-\lambda_n^{-ai} + \lambda_{n+1}^{-ai})
\]
\[
= \Gamma(1 + ia)\lambda_{n+1}^{-ai}(1 - \mu_n^{-ai}).
\]
But, for sufficiently large \( n, \mu_n > 1/M \). Hence by Lemma 8,
\[
\left| \int_0^\infty \phi(x)d(\Delta e^{-\lambda_n x}) \right| \geq C\Gamma(1 + ia)(1 - \mu_n).
\]
It therefore follows from (4.6) that $\sum_{n=0}^{\infty} |\varepsilon_n|(1 - \mu_n) < \infty$ as required.

5. The sufficiency part of the theorem can be improved. If (2.3) and (2.4) hold, then, in fact, $\sum_{n=0}^{\infty} a_n \varepsilon_n$ is summable $|R, \lambda_n, 1|$ whenever $\sum_{n=0}^{\infty} a_n$ is convergent. By Lemma 10 this is stronger than the present result. For summability $|R, \lambda_n, 1|$ implies summability $|A, \lambda_n|$ when the $(A, \lambda_n)$ method is applicable, i.e., whenever

$$\sum_{n=0}^{\infty} a_n \varepsilon_n e^{-\lambda x}$$

(5.1)

converges for all $x > 0$. But (2.3) alone is enough to ensure that whenever $\sum_{n=0}^{\infty} a_n$ converges, $\sum_{n=0}^{\infty} a_n \varepsilon_n$ converges, and thus (5.1) certainly converges.

Hence our Theorem 1 can be improved in the following way.

**Theorem 2.** Conditions (2.3) and (2.4) are sufficient for $\sum_{n=0}^{\infty} a_n \varepsilon_n$ to be summable $|R, \lambda_n, 1|$ whenever $\sum_{n=0}^{\infty} a_n$ converges, and necessary for it to be summable $|A, \lambda_n|$ whenever $\sum_{n=0}^{\infty} a_n$ converges and $\lambda_{n+1} = O(\lambda_n)$.

**Proof.** In the light of Theorem 1, it is sufficient to show that $\sum_{n=0}^{\infty} a_n \varepsilon_n$ is summable $|R, \lambda_n, 1|$ whenever $\sum_{n=0}^{\infty} a_n$ converges. To prove our assertion it is enough, by Lemma 9, to consider the discontinuous Riesz means, so write

$$t_n = -\frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k) a_k \varepsilon_k.$$

Suppose that $\sum_{k=0}^{\infty} a_k$ converges and that (2.3) and (2.4) hold. Suppose without loss of generality that $a_0 = 0$. Then, for $n > 1$,

$$t_n - t_{n-1} = (1/\lambda_n - 1/\lambda_{n+1}) \sum_{k=1}^{n} \lambda_k a_k \varepsilon_k$$

$$= (1/\lambda_n - 1/\lambda_{n+1}) \left[ \sum_{k=1}^{n-1} S_k \lambda_{k+1} - \lambda_k + \sum_{k=1}^{n-1} S_k \varepsilon_k \Delta \lambda_k + S_n \varepsilon_n \lambda_n \right]$$

$$= b_n + c_n + d_n \quad (\text{say}).$$

Supposing that for all $n, |S_n| < K$, we have

$$\sum_{n=1}^{\infty} |b_n| < K \sum_{k=1}^{\infty} |\Delta \varepsilon_k| \lambda_{k+1} \sum_{n=k+1}^{\infty} \frac{1}{\lambda_{n+1}}$$

$$= K \sum_{k=1}^{\infty} |\Delta \varepsilon_k| < \infty \quad \text{by (2.3)}.$$

$$\sum_{n=1}^{\infty} |c_n| < K \sum_{k=1}^{\infty} |\varepsilon_k| |\Delta \lambda_k| \sum_{n=k+1}^{\infty} \frac{1}{\lambda_{n+1}}$$

$$= K \sum_{k=1}^{\infty} |\varepsilon_k|/(\lambda_{k+1} - \lambda_k)/\lambda_{k+1}$$

$$= K \sum_{k=1}^{\infty} |\varepsilon_k|/(1 - \lambda_k/\lambda_{k+1}) = K \sum_{k=1}^{\infty} |\varepsilon_k|(1 - \mu_k) < \infty \quad \text{by (2.4)}.$$
\[ \sum_{n=1}^{\infty} |d_n| < K \sum_{n=1}^{\infty} |\varepsilon_n| \lambda_n (1/\lambda_n - 1/\lambda_{n+1}) \]
\[ = K \sum_{n=1}^{\infty} |\varepsilon_n| (1 - \lambda_n/\lambda_{n+1}) = K \sum_{n=1}^{\infty} |\varepsilon_n| (1 - \mu_n) \]
\[ < \infty \quad \text{by (2.4)}. \]

Hence the result.

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