PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 70, Number 1, June 1978

COMPOSITION OPERATOR ON l^p AND ITS ADJOINT

R. K. SINGH AND B. S. KOMAL

ABSTRACT. A necessary and sufficient condition for the invertibility of a composition operator C_ϕ on l^p is reported in this paper. The adjoint of C_ϕ is computed in the case $p = 2$. The necessary and sufficient conditions for unitary operators and co-isometries to be composition operators are also investigated. A study of invariant subspaces and reducing subspaces of C_ϕ is also made.

1. Preliminaries. If $(X, \mathcal{S}, \lambda)$ is a σ-finite measure space and ϕ is a nonsingular measurable transformation on X into itself, then the composition transformation C_ϕ on $L^p(\lambda)$ is defined by

$$C_\phi f = f \circ \phi \quad \text{for every } f \in L^p(\lambda).$$

If $C_\phi \in B(L^p(\lambda))$, the Banach algebra of all bounded linear operators on $L^p(\lambda)$, then it is called a composition operator induced by ϕ.

In this paper we are interested in the study of composition operators when X is equal to N, the set of all natural numbers and λ is the counting measure on N. In this case $L^p(\lambda)$ is equal to l^p.

2. Invertible composition operators on l^p.

THEOREM 2.1. *Let ϕ be a mapping on N into itself. Then $C_\phi \in B(l^p)$ if and only if there exists an integer $M > 0$ such that $\lambda(\phi^{-1}(\{n\})) \leqslant M$ for every $n \in N$. In the case $C_\phi \in B(l^p)$,*

$$\|C_\phi\|^p = \inf\left\{M: \lambda\left(\phi^{-1}(\{n\})\right) \leqslant M \text{ for all } n \in N\right\}.$$

PROOF. The proof of the first conclusion follows from Theorem 1 of **[6]**. To prove the second conclusion take $n \in N$. Then

$$\lambda\left(\phi^{-1}(\{n\})\right) = \|C_\phi e^{(n)}\|^p \leqslant \|C_\phi\|^p \|e^{(n)}\|^p = \|C_\phi\|^p,$$

where $e^{(n)}$ is the sequence defined by $e^{(n)}(m) = \delta_{mn}$ (the Kronecker delta). Since this is true for all $n \in N$, we get

$$\inf\left\{M: \lambda\left(\phi^{-1}(\{n\})\right) \leqslant M \text{ for all } n \in N\right\} \leqslant \|C_\phi\|^p.$$

For the reverse inequality, suppose $\lambda(\phi^{-1}(\{n\})) \leqslant M$ for all n in N. Then it is

Received by the editors November 26, 1976 and, in revised form, May 25, 1977 and September 21, 1977.

AMS (MOS) subject classifications (1970). Primary 47B99; Secondary 47B99.

Key words and phrases. Composition operator, invertible and unitary operators, isometry, co-isometry, adjoint, invariant and reducing subspaces.

clear that

$$\|C_\phi f\|^p \leqslant M\|f\|^p \quad \text{for all } f \in l^p.$$

Hence $\|C_\phi\|^p \leqslant \inf\{M\colon \lambda(\phi^{-1}(\{n\})) \leqslant M \text{ for all } n \in N\}$. Thus the proof is complete.

COROLLARY. *Let ϕ be a mapping from N into itself. Then ϕ is one-to-one if and only if $C_\phi \in B(l^p)$ and $\|C_\phi\| = 1$.*

PROOF. The proof follows from the last conclusion of Theorem 2.1.

COROLLARY. *If $C_\phi \in B(l^p)$, then the range of ϕ has infinitely many elements.*

PROOF. Proof follows from Theorem 2.1.

The invertibility of ϕ and the inducibility of the composition operator by the inverse of ϕ are necessary and sufficient conditions for the invertibility of C_ϕ on $L^p(\lambda)$, where λ is a σ-finite measure on a standard Borel space as is proved in [**5**]. In the case of l^p the invertibility of ϕ alone is a necessary and sufficient condition for the invertibility of C_ϕ.

This is shown in the following theorem.

THEOREM 2.2. *Let $C_\phi \in B(l^p)$. Then C_ϕ is invertible if and only if ϕ is invertible.*

PROOF. Suppose ϕ is invertible. Then there exists a mapping ψ such that $(\phi \circ \psi)(n) = (\psi \circ \phi)(n) = n$ for every $n \in N$. Since ψ is one-to-one, $C_\psi \in B(l^p)$ and $C_\psi C_\phi = C_\phi C_\psi = I$, the identity operator. Hence C_ϕ is invertible.

Conversely, suppose C_ϕ is invertible. If ϕ is not one-to-one, then $\phi(n) = \phi(m)$ for some distinct integers m and n and $x_n = x_m$ for all x in the range of C_ϕ. Thus C_ϕ is not onto, which is a contradiction. If ϕ is not onto, then there exists a positive integer k such that $k \notin \phi(N)$. An easy computation shows that $C_\phi e^{(k)} = 0$. Thus C_ϕ is not one-to-one, which is again a contradiction.

THEOREM 2.3. *Let $C_\phi \in B(l^2)$. Then the following are equivalent*:

(i) *C_ϕ is invertible*,

(ii) *C_ϕ is unitary*,

(iii) *C_ϕ is an isometry*.

PROOF. (i) implies (ii): If C_ϕ is invertible, then by Theorem 2.2, ϕ is invertible. If $x \in l^2$, then x and $x \circ \phi$ have same ranges and hence by [**1**, p. 70]

$$\sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} |x_{\phi(n)}|^2$$

i.e. $\|x\| = \|C_\phi x\|$. Thus C_ϕ is unitary.

(ii) implies (iii): The proof is trivial.

(iii) implies (i): Suppose C_ϕ is an isometry.

Then ϕ is onto because if ϕ is not onto, then that C_ϕ has a nontrivial kernel is a contradiction. Also if ϕ is not one-to-one, then $\|C_\phi\| > 1$ which is again a contradiction. Hence ϕ is invertible.

COROLLARY. *Let $C_\phi \in B(l^2)$. Then C_ϕ^* is a composition operator if and only if C_ϕ is invertible.*

PROOF. Suppose C_ϕ is invertible. Then by Theorem 2.3, C_ϕ is unitary. Hence $C_\phi^* C_\phi = C_\phi C_\phi^* = I$, which shows that $C_\phi^* = C_\phi^{-1} = C_{\phi^{-1}}$ (ϕ^{-1} denotes the inverse of ϕ). Thus C_ϕ^* is a composition operator.

Conversely, if C_ϕ^* is a composition operator, then $C_\phi^* = C_\psi$ for some ψ. If X_E denotes the characteristic function of the set E, then for every $n \in N$,

$$X_{\{\phi(n)\}} = C_\phi^* X_{\{n\}} = C_\psi X_{\{n\}} = X_{\psi^{-1}(\{n\})}.$$

Hence $\{\phi(n)\} = \psi^{-1}(\{n\})$ for all n in N. This guarantees the invertibility of ψ, and therefore C_ψ is invertible by Theorem 2.2. This shows that C_ϕ is invertible.

3. The adjoint of C_ϕ. Let ϕ be a mapping on N into itself such that $C_\phi \in B(l^2)$. Then we define a transformation T on l^2 by

$$(Tx)(k) = \langle Tx, e^{(k)} \rangle = \langle x, C_\phi e^{(k)} \rangle = \langle x, X_{\phi^{-1}(\{k\})} \rangle,$$

where $X_{\phi^{-1}(\{k\})}$ denotes the characteristic function of $\phi^{-1}(\{k\})$.

$$= \begin{cases} \sum_{n \in \phi^{-1}(\{k\})} x_n & \text{if } \phi^{-1}(\{k\}) \text{ is nonempty,} \\ 0 & \text{if } \phi^{-1}(\{k\}) \text{ is empty.} \end{cases}$$

A simple computation shows that T is the adjoint of C_ϕ, i.e., $T = C_\phi^*$.

THEOREM 3.1. *Let $C_\phi \in B(l^2)$. Then the following are equivalent:*
(i) *C_ϕ is onto,*
(ii) *C_ϕ is a co-isometry,*
(iii) *C_ϕ is a partial isometry.*

PROOF. (i) implies (ii): Suppose C_ϕ is onto. Then by the proof of the 'only if' part of Theorem 1.2, ϕ is one-to-one and hence by definition of C_ϕ^*, $\|C_\phi^* x\| = \|x\|$ for every $x \in l^2$. Thus C_ϕ is a co-isometry.

(ii) implies (iii): If C_ϕ is a co-isometry, then C_ϕ^* is a partial isometry and hence by [**2**, p. 96] C_ϕ is a partial isometry.

(iii) implies (i): Suppose C_ϕ is a partial isometry. Then ϕ is one-to-one. For, if ϕ is not one-to-one, then $\|C_\phi\| > 1$ by the corollary to Theorem 2.1, which is a contradiction. In view of the proof of Theorem 2 [**4**], C_ϕ has dense range, and since the range of every partial isometry is closed, we can conclude that C_ϕ is onto.

4. Unitary composition operators.

THEOREM 4.1. *Let A be a unitary operator on l^2. Then A is a composition operator if and only if for every $n \in N$ there exists an $m \in N$ such that $Ae^{(n)} = e^{(m)}$.*

PROOF. Suppose the condition is true. Then since A is unitary, $\{Ae^{(n)}: n \in N\}$ is a maximal orthonormal family contained in $\{e^{(m)}: m \in N\}$ and

hence $\{Ae^{(n)}: n \in N\} = \{e^{(m)}: m \in N\}$. Let $m \in N$. Then there exists an $n \in N$ such that $Ae^{(n)} = e^{(m)}$. Define $\phi(m) = n$. Then ϕ is a well-defined surjection on N and $Ae^{(n)} = e^{(m)} = e^{\phi^{-1}}(n) = C_\phi e^{(n)}$. Thus $A = C_\phi$.

The converse is obvious.

THEOREM 4.2. *Let A be a co-isometry on l^2. Then A is a composition operator if and only if $\{A^*e^{(n)}: n \in N\} \subset \{e^{(m)}: m \in N\}$.*

PROOF. Suppose that A is a composition operator. Then $A = C_\phi$ and $\{A^*e^{(n)}: n \in N\} = \{C_\phi^* e^{(n)}: n \in N\} = \{e^{\phi(n)}: n \in N\} \subset \{e^{(m)}: m \in N\}$.

Conversely, suppose that the condition is true. Then for every $n \in N$ there exists a unique $m \in N$ such that $A^*e^{(n)} = e^{(m)}$. Define $\phi(n) = m$. Then ϕ is well defined and is one-to-one since A^* is one-to-one. Let C_ϕ be a composition operator induced by ϕ. Then $A^*e^{(n)} = C_\phi^* e^{(n)}$ and hence $A^* = C_\phi^*$. Thus $A = C_\phi$.

5. Invariant and reducing subspaces of composition operators.

THEOREM 5.1. *Every composition operator on l^2 has an invariant subspace.*

PROOF. Suppose C_ϕ is a composition operator on l^2. Then if C_ϕ is invertible, then by Theorem 2.3, C_ϕ is unitary and therefore it is normal. Hence C_ϕ has an invariant subspace. If C_ϕ is not invertible, then either C_ϕ is not onto in which case range of C_ϕ is invariant under C_ϕ **[8]** or C_ϕ is not one-to-one in which case kernel of C_ϕ is invariant under C_ϕ. Thus the proof is complete.

DEFINITION. Two integers are in the same orbit of ϕ if each can be reached from the other by composing ϕ and ϕ^{-1} sufficiently many times, where ϕ^{-1} means a multivalued function.

THEOREM 5.2. *Let ϕ be a function on N into itself such that $C_\phi \in B(l^2)$. Then C_ϕ has a reducing subspace if there exist two distinct elements of N which are not in the same orbit of ϕ.*

PROOF. Let m_0 and n_0 be two distinct elements which are not in the same orbit and let $E = \{n: n \text{ and } n_0 \text{ are not in the same orbit of } \phi\}$.

Suppose $M = \text{Span}\{e^{(n)}: n \in E\}$. Then M is a proper closed subspace of l^2.

For every $n \in E$, $C_\phi e^{(n)}$ and $C_\phi^* e^{(n)}$ belong to M, and therefore M is invariant under C_ϕ and C_ϕ^*. Hence M is a reducing subspace of C_ϕ.

REFERENCES

1. T. J. I'A. Bromwich, *An introduction to the theory of infinite series*, Macmillan, New York, 1926.

2. R. G. Douglas, *Banach algebra techniques in operator theory*, Academic Press, New York, 1972.

3. P. R. Halmos, *A Hilbert space problem book*, Springer-Verlag, Berlin and New York, 1974.

4. R. K. Singh, *Normal and Hermitian composition operators*, Proc. Amer. Math. Soc. **47** (1975), 348–350.

5. _____, *Invertible composition operators on* $L^2(\lambda)$, Proc. Amer. Math. Soc. **56** (1976), 127–129.

6. _____, *Composition operators induced by rational functions*, Proc. Amer. Math. Soc. **59** (1976), 329–333.

7. _____, *Composition operators*, Doctoral Thesis, University of New Hampshire, 1972.

8. R. K. Singh and Ashok Kumar, *Multiplication operators and composition operators with closed ranges*, Bull. Austral. Math. Soc. **16** (1977), 247–252.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF JAMMU, JAMMU 180001 INDIA