COMPOSITION OPERATOR ON $l^p$ AND ITS ADJOINT

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Abstract. A necessary and sufficient condition for the invertibility of a composition operator $C_\phi$ on $l^p$ is reported in this paper. The adjoint of $C_\phi$ is computed in the case $p = 2$. The necessary and sufficient conditions for unitary operators and co-isometries to be composition operators are also investigated. A study of invariant subspaces and reducing subspaces of $C_\phi$ is also made.

1. Preliminaries. If $(X, \mathcal{S}, \lambda)$ is a $\sigma$-finite measure space and $\phi$ is a nonsingular measurable transformation on $X$ into itself, then the composition transformation $C_\phi$ on $L^p(X)$ is defined by

$$C_\phi f = f \circ \phi \text{ for every } f \in L^p(X).$$

If $C_\phi \in B(L^p(\lambda))$, the Banach algebra of all bounded linear operators on $L^p(\lambda)$, then it is called a composition operator induced by $\phi$.

In this paper we are interested in the study of composition operators when $X$ is equal to $N$, the set of all natural numbers and $\lambda$ is the counting measure on $N$. In this case $L^p(\lambda)$ is equal to $l^p$.

2. Invertible composition operators on $l^p$.

Theorem 2.1. Let $\phi$ be a mapping on $N$ into itself. Then $C_\phi \in B(l^p)$ if and only if there exists an integer $M > 0$ such that $\lambda(\phi^{-1}(\{n\})) \leq M$ for every $n \in N$. In the case $C_\phi \in B(l^p)$,

$$\|C_\phi\|^p = \inf \{ M : \lambda(\phi^{-1}(\{n\})) \leq M \text{ for all } n \in N \}.$$  

Proof. The proof of the first conclusion follows from Theorem 1 of [6]. To prove the second conclusion take $n \in N$. Then

$$\lambda(\phi^{-1}(\{n\})) = \|C_\phi e^{(n)}\|^p \leq \|C_\phi\|\|e^{(n)}\|^p = \|C_\phi\|^p,$$

where $e^{(n)}$ is the sequence defined by $e^{(n)}(m) = \delta_{mn}$ (the Kronecker delta). Since this is true for all $n \in N$, we get

$$\inf \{ M : \lambda(\phi^{-1}(\{n\})) \leq M \text{ for all } n \in N \} \leq \|C_\phi\|^p.$$  

For the reverse inequality, suppose $\lambda(\phi^{-1}(\{n\})) \leq M$ for all $n$ in $N$. Then it is
clear that
\[ \|C_\phi f\|_p \leq M\|f\|_p \quad \text{for all } f \in L^p. \]
Hence \( \|C_\phi\|^p \leq \inf\{M : \lambda(\phi^{-1}(\{n\})) < M \text{ for all } n \in N\} \). Thus the proof is complete.

**Corollary.** Let \( \phi \) be a mapping from \( N \) into itself. Then \( \phi \) is one-to-one if and only if \( C_\phi \in B(L^p) \) and \( \|C_\phi\| = 1 \).

**Proof.** The proof follows from the last conclusion of Theorem 2.1.

**Corollary.** If \( C_\phi \in B(L^p) \), then the range of \( \phi \) has infinitely many elements.

**Proof.** Proof follows from Theorem 2.1.

The invertibility of \( \phi \) and the inducibility of the composition operator by the inverse of \( \phi \) are necessary and sufficient conditions for the invertibility of \( C_\phi \) on \( L^p(\lambda) \), where \( \lambda \) is a \( \sigma \)-finite measure on a standard Borel space as is proved in [5]. In the case of \( L^p \) the invertibility of \( \phi \) alone is a necessary and sufficient condition for the invertibility of \( C_\phi \).

This is shown in the following theorem.

**Theorem 2.2.** Let \( C_\phi \in B(L^p) \). Then \( C_\phi \) is invertible if and only if \( \phi \) is invertible.

**Proof.** Suppose \( \phi \) is invertible. Then there exists a mapping \( \psi \) such that \((\phi \circ \psi)(n) = (\psi \circ \phi)(n) = n \) for every \( n \in N \). Since \( \psi \) is one-to-one, \( C_\psi \in B(L^p) \) and \( C_\phi C_\psi = C_\psi C_\phi = I \), the identity operator. Hence \( C_\phi \) is invertible.

Conversely, suppose \( C_\phi \) is invertible. If \( \phi \) is not one-to-one, then \( \phi(n) = \phi(m) \) for some distinct integers \( m \) and \( n \) and \( x_n = x_m \) for all \( x \) in the range of \( C_\phi \). Thus \( C_\phi \) is not onto, which is a contradiction. If \( \phi \) is not onto, then there exists a positive integer \( k \) such that \( k \notin \phi(N) \). An easy computation shows that \( C_\phi e^{(k)} = 0 \). Thus \( C_\phi \) is not one-to-one, which is again a contradiction.

**Theorem 2.3.** Let \( C_\phi \in B(l^2) \). Then the following are equivalent:

(i) \( C_\phi \) is invertible,
(ii) \( C_\phi \) is unitary,
(iii) \( C_\phi \) is an isometry.

**Proof.** (i) implies (ii): If \( C_\phi \) is invertible, then by Theorem 2.2, \( \phi \) is invertible. If \( x \in l^2 \), then \( x \) and \( x \circ \phi \) have same ranges and hence by [1, p. 70]
\[ \sum_{n=1}^{\infty} |x_n|^2 = \sum_{n=1}^{\infty} |x_{\phi(n)}|^2 \]
i.e. \( \|x\| = \|C_\phi x\| \). Thus \( C_\phi \) is unitary.

(ii) implies (iii): The proof is trivial.

(iii) implies (i): Suppose \( C_\phi \) is an isometry.

Then \( \phi \) is onto because if \( \phi \) is not onto, then that \( C_\phi \) has a nontrivial kernel is a contradiction. Also if \( \phi \) is not one-to-one, then \( \|C_\phi\| > 1 \) which is again a contradiction. Hence \( \phi \) is invertible.
Corollary. Let \( C_\phi \in B(l^2) \). Then \( C_\phi^* \) is a composition operator if and only if \( C_\phi \) is invertible.

Proof. Suppose \( C_\phi \) is invertible. Then by Theorem 2.3, \( C_\phi \) is unitary. Hence \( C_\phi^* C_\phi = \phi^{-1} \), which shows that \( C_\phi^* = \phi^{-1} \) (\( \phi^{-1} \) denotes the inverse of \( \phi \)). Thus \( C_\phi^* \) is a composition operator.

Conversely, if \( C_\phi^* \) is a composition operator, then \( C_\phi^* = \phi \) for some \( \psi \). If \( X_E \) denotes the characteristic function of the set \( E \), then for every \( n \in \mathbb{N} \),

\[
X(\phi(n)) = C_\phi^* X(n) = C_\psi X(n) = X(\psi^{-1}(n))
\]

Hence \( \{\phi(n)\} = \psi^{-1}(\{n\}) \) for all \( n \in \mathbb{N} \). This guarantees the invertibility of \( \psi \), and therefore \( C_\psi \) is invertible by Theorem 2.2. This shows that \( C_\phi \) is invertible.

3. The adjoint of \( C_\phi \). Let \( \phi \) be a mapping on \( \mathbb{N} \) into itself such that \( C_\phi \in B(l^2) \). Then we define a transformation \( T \) on \( l^2 \) by

\[
(Tx)(k) = \langle Tx, e^{(k)} \rangle = \langle x, C_\phi e^{(k)} \rangle = \langle x, X(\phi^{-1}((k))) \rangle
\]

where \( X(\phi^{-1}((k))) \) denotes the characteristic function of \( \phi^{-1}(\{k\}) \).

A simple computation shows that \( T \) is the adjoint of \( C_\phi \), i.e., \( T = C_\phi^* \).

Theorem 3.1. Let \( C_\phi \in B(l^2) \). Then the following are equivalent:

(i) \( C_\phi \) is onto,
(ii) \( C_\phi \) is a co-isometry,
(iii) \( C_\phi \) is a partial isometry.

Proof. (i) implies (ii): Suppose \( C_\phi \) is onto. Then by the proof of the 'only if' part of Theorem 1.2, \( \phi \) is one-to-one and hence by definition of \( C_\phi^* \), \( ||C_\phi^* x|| = ||x|| \) for every \( x \in l^2 \). Thus \( C_\phi \) is a co-isometry.

(ii) implies (iii): If \( C_\phi \) is a co-isometry, then \( C_\phi^* \) is a partial isometry and hence by [2, p. 96] \( C_\phi \) is a partial isometry.

(iii) implies (i): Suppose \( C_\phi \) is a partial isometry. Then \( \phi \) is one-to-one. For, if \( \phi \) is not one-to-one, then \( ||C_\phi|| > 1 \) by the corollary to Theorem 2.1, which is a contradiction. In view of the proof of Theorem 2 [4], \( C_\phi \) has dense range, and since the range of every partial isometry is closed, we can conclude that \( C_\phi \) is onto.

4. Unitary composition operators.

Theorem 4.1. Let \( A \) be a unitary operator on \( l^2 \). Then \( A \) is a composition operator if and only if for every \( n \in \mathbb{N} \) there exists an \( m \in \mathbb{N} \) such that \( A e^{(n)} = e^{(m)} \).

Proof. Suppose the condition is true. Then since \( A \) is unitary, \( \{A e^{(n)} : n \in \mathbb{N}\} \) is a maximal orthonormal family contained in \( \{e^{(m)} : m \in \mathbb{N}\} \) and
hence \( \{Ae(n): n \in N\} = \{e(m): m \in N\} \). Let \( m \in N \). Then there exists an \( n \in N \) such that \( Ae(n) = e(m) \). Define \( \phi(n) = n \). Then \( \phi \) is a well-defined surjection on \( N \) and \( Ae(n) = e(m) = e^{\phi^{-1}}(n) = C_{\phi}e(n) \). Thus \( A = C_{\phi} \).

The converse is obvious.

**Theorem 4.2.** Let \( A \) be a co-isometry on \( l^2 \). Then \( A \) is a composition operator if and only if \( \{A*e(n): n \in N\} \subseteq \{e(m): m \in N\} \).

**Proof.** Suppose that \( A \) is a composition operator. Then \( A = C_{\phi} \) and \( \{A*e(n): n \in N\} = \{C_{\phi}*e(n): n \in N\} = \{e^{\phi(n)}(n): n \in N\} \subseteq \{e(m): m \in N\} \).

Conversely, suppose that the condition is true. Then for every \( n \in N \) there exists a unique \( m \in N \) such that \( A*e(m) = e(m) \). Define \( \phi(n) = m \). Then \( \phi \) is well defined and is one-to-one since \( A^* \) is one-to-one. Let \( C_{\phi} \) be a composition operator induced by \( \phi \). Then \( A*e(n) = C_{\phi}*e(n) \) and hence \( A^* = C_{\phi}^* \). Thus \( A = C_{\phi} \).

5. Invariant and reducing subspaces of composition operators.

**Theorem 5.1.** Every composition operator on \( l^2 \) has an invariant subspace.

**Proof.** Suppose \( C_{\phi} \) is a composition operator on \( l^2 \). Then if \( C_{\phi} \) is invertible, then by Theorem 2.3, \( C_{\phi} \) is unitary and therefore it is normal. Hence \( C_{\phi} \) has an invariant subspace. If \( C_{\phi} \) is not invertible, then either \( C_{\phi} \) is not onto in which case range of \( C_{\phi} \) is invariant under \( C_{\phi} \) [8] or \( C_{\phi} \) is not one-to-one in which case kernel of \( C_{\phi} \) is invariant under \( C_{\phi} \). Thus the proof is complete.

**Definition.** Two integers are in the same orbit of \( \phi \) if each can be reached from the other by composing \( \phi \) and \( \phi^{-1} \) sufficiently many times, where \( \phi^{-1} \) means a multivalued function.

**Theorem 5.2.** Let \( \phi \) be a function on \( N \) into itself such that \( C_{\phi} \in B(l^2) \). Then \( C_{\phi} \) has a reducing subspace if there exist two distinct elements of \( N \) which are not in the same orbit of \( \phi \).

**Proof.** Let \( m_0 \) and \( n_0 \) be two distinct elements which are not in the same orbit and let \( E = \{n: n \text{ and } n_0 \text{ are not in the same orbit of } \phi\} \).

Suppose \( M = \text{Span}\{e(n): n \in E\} \). Then \( M \) is a proper closed subspace of \( l^2 \).

For every \( n \in E \), \( C_{\phi}e(n) \) and \( C_{\phi}^*e(n) \) belong to \( M \), and therefore \( M \) is invariant under \( C_{\phi} \) and \( C_{\phi}^* \). Hence \( M \) is a reducing subspace of \( C_{\phi} \).

**REFERENCES**


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