A REMARK ON THE GROTHENDIECK RESIDUE MAP

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Abstract. The purpose of this note is to give a direct proof that a global integral over a compact complex manifold $X$ can be evaluated on the zero set of a meromorphic vector field on $X$ with isolated zeros via a Grothendieck residue morphism. A special case of this evaluation is the meromorphic vector field theorem of Baum and Bott [1]. The present proof suggests some complements of the M.V.F. Theorem which are contained in Theorem 2.

1. Introduction. Statement of results. Given a finite, nontrivial, but possibly unreduced subvariety $Z$ of a connected compact complex manifold $X$ of dimension $n$, and given $\omega \in H^n(X, \Omega^n)$, the global integral $\int_X \omega$ can always be evaluated as a sum of residues (in the sense of Grothendieck) on $Z$. The proofs of this fact in the literature are valid in algebraic geometry and are necessarily complicated, cf. [7], [13]. On the other hand, what we shall show is that if one makes the assumption that $Z$ is the variety of zeros of a meromorphic vector field $V$ on $X$, then a reformulation of the fundamental commutative diagram of [13], which makes the contribution from $Z$ explicit, can be proven simply (Theorem 1). Moreover this approach immediately suggests some complements to the Meromorphic Vector Field Theorem [1], [2], which are given in Theorem 2.

To state our results precisely, let $T$ denote the holomorphic tangent bundle of $X$, $L$ a holomorphic line bundle on $X$, and $\Theta$ (resp. $\mathcal{L}$) the sheaf of germs of holomorphic sections of $T$ (resp. $L$). Contraction by $V \in H^0(X, T \otimes L)$ on holomorphic $p$-forms is an operator $i(V): \Omega^p \rightarrow \Omega^{p-1} \otimes \mathcal{L}$ on the sheaf level, which defines a sheaf of ideals $I_Z = i(V)(\mathcal{O}_X \otimes \mathcal{L}^{-1})$ in $\mathcal{O}^0 = \mathcal{O}$. The subvariety $Z$ defined by $I_Z$ is called the variety of zeros of $V$. The structure sheaf of $Z$ is by definition the sheaf of rings $\mathcal{O}_Z = \mathcal{O} / I_Z$. For any sheaf $\mathcal{F}$ of $\mathcal{O}$-modules on $X$, let $\mathcal{F}_Z = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_Z$. The Grothendieck Residue Map we will associate to $V$ is a map $\text{Res}: H^0(X, \mathcal{L}_Z^n) \rightarrow \mathbb{C}$ about which we shall prove the following result. It will always be assumed that $Z$ is finite but nontrivial.

**Theorem 1.** There exists a map $m: H^0(X, \mathcal{L}_Z^n) \rightarrow H^n(X, \Omega^n)$ (depending only on $V$) such that

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commutes, where \( \text{tr} \) is the map \((1/2\pi i)^n f_X\) (cf. [13]).

Given an ad-invariant linear map \( p: \text{gl}(n, C)^{\otimes k} \rightarrow C \), there is a natural element \( p(V_0) \) of \( H^0(X, \mathcal{L}_2) \) such that in the case \( k = n \), \( \text{tr}(m(p(V_0))) \) is a characteristic number of the virtual bundle \( \mathcal{T} - L^{-1} \). The assertion 
\((-1)^m \text{tr}(m(p(V_0))) = \text{Res} p(V_0) \) is of course the M.V.F. Theorem. In §5, we will show that in addition one can use (1) to prove

**THEOREM 2.** Suppose \( \text{deg} p = k < n \) and that \( \sigma \in H^0(X, L) \). Let \( \sigma^{n-k} p(V_0) \) denote the image of \( \sigma^{n-k} \otimes p(V_0) \) under the natural pairing \( H^0(X, L^{n-k}) \otimes H^0(X, \mathcal{L}_2) \rightarrow H^0(X, \mathcal{L}_2) \). Then \( \text{Res} \sigma^{n-k} p(V_0) = 0 \).

To prove Theorem 1, we employ a double complex with differentials \( i(V) \) and \( \partial \) to compute \( H^0(X, \mathcal{L}_2) \). The mapping \( m \) is an edge morphism in this double complex. (1) is a consequence of the projector trick of Bott [3] and an interesting local integral representation formula for \( \text{Res} \) (Lemma 4), which simply amounts to combining a local I.R.F. for the partial derivatives of a holomorphic function at a point with the computational algorithm for the local residue involving the Nullstellensatz.

2. **A double complex.** For a fixed integer \( m \), let \( A^{p,q}(L^m) \) denote \( C^\infty, L^m \)-valued forms on \( X \) of type \((p, q)\). The operators \( i(V): A^{p,q}(L^m) \rightarrow A^{p-1,q}(L^{m+1}) \) and \( \partial: A^{p,q}(L^m) \rightarrow A^{p+1,q}(L^m) \) satisfy \( i(V)^2 = \partial^2 = \partial i(V) + i(V)\partial = 0 \), so \( D = i(V) + \partial \) is a total differential for the complex \( \mathcal{C}_m = \Sigma_k C^{p,k} \) formed from the double complex \( \{ C^{p,k} = A^{p,q}(L^{m-k}), i(V), \partial \} \).

**LEMMA 1.** Assuming that \( Z \) is finite, then \( H^0(C_m) \cong H^0(X, \mathcal{L}_2) \).

Let \( \mathcal{E}^{p,q}(\mathcal{L}_k) \) denote the sheaf of germs of \( C^\infty, L^k \)-valued \((p, q)\) forms on \( X \). To prove Lemma 1, we need

**LEMMA 2. For any \( k \) we have a fine resolution of \( \mathcal{L}_k \):**

\[
0 \rightarrow \mathcal{L}_2 \xrightarrow{h} \mathcal{E}^{0,0}(\mathcal{L}_k)/i(V)\mathcal{E}^{1,0}(\mathcal{L}_k-1) \xrightarrow{\partial} \mathcal{E}^{0,1}(\mathcal{L}_k)/i(V)\mathcal{E}^{1,1}(\mathcal{L}_k-1) \xrightarrow{\partial} \cdots \xrightarrow{\partial} \mathcal{E}^{0,n}(\mathcal{L}_k)/i(V)\mathcal{E}^{1,n}(\mathcal{L}_k-1) \rightarrow 0
\]  
\[
(2)
\]

where \( h \) is the natural inclusion.

**PROOF.** One first notes that for any \( k \) the Dolbeault resolution

\[
0 \rightarrow \mathcal{L}_k \rightarrow \mathcal{E}^{0,0}(\mathcal{L}_k) \rightarrow \mathcal{E}^{0,1}(\mathcal{L}_k) \rightarrow \cdots \rightarrow \mathcal{E}^{0,n}(\mathcal{L}_k) \rightarrow 0
\]
is an exact sequence of \( \mathcal{O} \)-modules from which (2) is obtained via tensorisation by \( \mathcal{L}_2 \). Lemma 2 follows from the fact that \( \mathcal{E}^{p,q}(\mathcal{L}_k) \) is a flat \( \mathcal{O} \)-module,
by the Malgrange Preparation Theorem [9].

This lemma does not assume $Z$ is finite. Finiteness of $Z$ is used to conclude exactness of

$$0 \to \Omega^n \otimes \mathbb{C}^{k-n} \to \cdots \to \Omega^1 \otimes \mathbb{C}^{k-1} \to \mathbb{C}^k.$$  (3)

To prove Lemma 1, consider the exact sequence obtained from (2):

$$0 \to H^0(X, \mathbb{C}^k_z) \xrightarrow{h} C^0_{k,0} / i(V)C^1_{k,1} \xrightarrow{\delta} C^0_{k,1} / i(V)C^1_{k,1}.$$ (4)

For $s \in H^0(X, \mathbb{C}^k_z)$, choose $s_0 \in C^0_{k,0}$ such that $h(s) = s_0$ modulo $i(V)C^1_{k,1}$. By (4), $\delta s_0 = i(V)s_1$ for some $s_1 \in C^1_{k,1}$. Since $i(V)\delta s_1 = -\delta i(V)s_1 = \delta^2 s_0 = 0$, there exists by (3) an $s_2 \in C^1_{k,2}$ so that $\delta s_1 = i(V)s_2$. Continuing in this manner, one gets a total cocycle $s = s_0 - s_1 + \cdots + (-1)^n s_n$ of $C^*_k$ of degree 0, and the map $s \to s$ induces a morphism $\phi_s: H^0(X, \mathbb{C}^k_z) \to H^0(C_k)$. To show that $\phi_s$ is an isomorphism, we produce its inverse. Let $S$ be a total cocycle in $C^*_k$ denoted as above. Then $\delta s_0$ is in $i(V)C^1_{k,1}$. Hence by (4), there exists a unique $s \in H^0(X, \mathbb{C}^k_z)$ such that $h(s) = s_0 \mod i(V)C^1_{k,1}$. Then $S \to s$ induces the inverse of $\phi_s$, and the proof of Lemma 1 is complete.

One now defines $m: H^0(X, \mathbb{C}^n_z) \to H^n(X, \mathbb{C}^n)$ ($n = \dim X$) by composing $\phi_s^{-1}$ with the natural edge morphism $H^0(C^*_n) \to H^0(X, \mathbb{C}^n)$ induced by mapping $S$ in $C^*_n$ to the component $s_n$ of $S$ in $A^n_{\mathbb{C}^n}(L^0)$.

3. The morphism $\text{Res}$. Let $U$ be an open ball about the origin $O$ of $\mathbb{C}^n$, which is the only common zero of $a_1, \ldots, a_n \in H^0(U, \mathbb{C})$. The local residue at $O$ of $\omega \in H^0(U, \mathbb{C}^n)$ with respect to $a_1, \ldots, a_n$ is defined in [11] as

$$\text{Res}(a_1, \ldots, a_n) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial D \times \cdots \times \partial D} \frac{\omega}{a_1 \cdots a_n}$$ (5)

where $D$ is a disc about $O$ in $\mathbb{C}$ chosen so that $\partial D \times \cdots \times \partial D$ misses the hypersurface $\{z: (a_1, \ldots, a_n)(z) = 0\}$ in $U$. The local residue defined by (5) coincides with the Grothendieck residue, and hence can be computed by the well-known algorithm [1], [2], [7], [11], [13]. If $a_1, \ldots, a_n \in H^0(U, \mathbb{C})$, then

$$\text{Res}(a_1, \ldots, a_n)$$

is unambiguously defined for all $\omega \in H^0(U, \mathbb{C}^n)$.

Suppose $V \in H^0(X, T \otimes L)$ has isolated zeros $Z$, and let $(z_1, \ldots, z_n)$ denote local coordinates for $X$ near $\xi \in Z$ with $z_i(\xi) = 0$ for each $i$. One may locally express $V = \Sigma a_i \partial / \partial z_i$, where $a_1, \ldots, a_n$ are local sections of $L$ whose only common zero is $\xi$. Given $s \in \mathbb{C}^n_\mathbb{C}$, the local residue of $s$ at $\xi$ is defined to be

$$\text{Res}(s) = \text{Res}(s \frac{dz_1 \cdots dz_n}{a_1 \cdots a_n})$$ (6)

Under change of coordinates, $\text{Res}$ transforms in a manner so as to imply that (6) depends only on $V$ and $s$. In fact, $\text{Res}(s) = 0$ if $s \in (L\mathbb{C}^n)_\mathbb{C}$ (by [7]), and consequently $\text{Res}$ is defined as a morphism $\text{Res}_\xi: \mathbb{C}^n_{\mathbb{C}_{\mathbb{C}}} \to C$. Define $\text{Res}$:

$$H^0(X, \mathbb{C}^n_z) \to C$$ by $\text{Res} = \Sigma \text{Res}_\xi$. 

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4. The proof of Theorem 1. In order to localize \( \omega = m(S) \in H^\ast(X, \Omega^n) \), we will use Bott's projector trick. Recall that projector for \( V \) is an \( L^{-1} \) valued \((1, 0)\) form \( \pi \) on \( X - Z \) such that \( \pi(V) = 1 \). Bott's key observation concerning projectors can be rephrased in this context as the observation that the differential form

\[
\tau = \pi\left(s_0(\bar{\partial} \pi)^{n-1} - s_1(\bar{\partial} \pi)^{n-2} + \cdots + (-1)^{n-1} s_{n-1}\right)
\]

on \( X - Z \) satisfies \( s_n = (-1)^{n-1} \, d\tau \) provided \( S = s_0 + s_1 + \cdots + s_n \) is a total cocycle of \( C^0_n \). If \( \{W_j\} \) is a finite covering of \( Z \) by disjoint coordinate balls such that \( W_i \cap Z = \{z_i\} \), then by Stokes' Theorem

\[
\int_{X} s_n = (-1)^n \sum \int_{\partial W_j} \tau.
\]

Now the right-hand side of (8) can be vastly simplified by the following observation.

**Lemma 3.** For any \( S \in H^0(C) \), there is a representing cocycle \( S = s_0 + \cdots + s_n \) such that \( s_0 \) is holomorphic near \( Z \), and, if \( i > 0 \), then \( s_i = 0 \) near \( Z \).

**Proof.** \( s_0 \) can obviously be so chosen. The existence of the \( g_i \) follows from repeated application of (3).

The proof of Theorem 1 now results from the following local integral representation formula for \( \text{Res} \).

**Lemma 4.** For any projector \( \pi \) for \( V \) and for any \( \xi \in Z \),

\[
\text{Res}(s) \xi = \left(\frac{1}{2\pi i}\right)^n \int_{\partial W} s\pi(\bar{\partial} \pi)^{n-1}
\]

where \( s \in \mathcal{L}^n_\xi \) and \( W \) is a sufficiently small ball centered at \( \xi \).

**Proof.** Since \( Z \) is finite, \( \mathcal{L}^n_\xi \subset H^0(X, \Omega^n_\xi) \). Therefore, \( m(s) \) is defined and \( \int m(s) = \int_{\partial W} s\pi(\bar{\partial} \pi)^{n-1} \). Hence (9) is independent of \( \pi \). Let \( \pi \) be the projector for \( V \) defined in a neighborhood \( W \) of \( \xi \) as follows. Let \( V = \sum a_i \partial/\partial z_i \) on \( W \) as above. By Hilbert's Nullstellensatz, there exist positive integers \( \alpha_1, \ldots, \alpha_n \) and \( b_{ij} \in H^0(W, L^{-1}) \) such that \( z_i^{\alpha_i} = \Sigma b_j a_j \). Set \( \pi = u^{-1} \Sigma z_i^{\alpha_i} b_j dz_j \), where \( u = \Sigma(z_i z_i)^{\alpha_i} \). Note

\[
\pi(\bar{\partial} \pi)^{n-1} = (n-1)! \left( (-1)^{n(n-1)/2} \det \|b_j\| \right)
\]

\[
\times \sum (-1)^{i-1} z_i^{\alpha_i} dz_i^{\alpha_i} \cdots \cdots dz_i^{\alpha_i} \wedge \cdots \wedge dz_i^{\alpha_i} \wedge dz_1 \wedge \cdots \wedge dz_n.
\]

Now, by applying a standard I.R.F. for the partial derivatives of a holomorphic function \( g \) defined in a neighbourhood of \( \bar{W} \) [10, p. 56], one gets

\[
\left(\frac{1}{2\pi i}\right)^n \int_{\partial W} g\pi(\bar{\partial} \pi)^{n-1} = 1 / (\alpha - 1)! D^\alpha (\det \|b_j\| g)(\xi)
\]

where \( \alpha \) and \( \alpha - 1 \) denote, respectively, the multi-indices \( (\alpha_1, \ldots, \alpha_n) \) and \( (\alpha_1 - 1, \ldots, \alpha_n - 1) \). Lemma 4 follows from the fact that the r.h.s. of (10) is
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\[ \text{Res}(g \frac{dz_1 \cdots dz_n}{a_1, \ldots, a_n}) \]

by the algorithm.

To finish the proof of (1), let \( s \in H^0(X, \mathcal{E}_Z^n) \) and let \( S = s_0 + s_1 + \cdots + s_n \) be a total cocycle of \( \mathcal{E}_Z^n \) representing \( \phi(s) \) satisfying all the conclusions of Lemma 3. Then \((-1)^n m(S) = (-1)^n s_n\), so

\[ (-\frac{1}{2\pi i})^n \int_X s_n = \left( \frac{1}{2\pi i} \right)^n \sum \int_{\partial W_j} s_0 \pi (\tilde{\pi})^{n-1} = \text{Res}(s) \]

by (9) and Lemma 4.

5. Some residue formulas. If \( V \) is a global section of the tangent sheaf \( \Theta \) of \( X \) (i.e. a holomorphic vector field on \( X \)), then the Lie bracket \( Y \rightarrow [V, Y] \) induces a \( C \)-linear map \( \tilde{V}: \Theta \rightarrow \Theta \) lifting the derivation \( V: \Theta \rightarrow \Theta \), i.e. \( \tilde{V}(fY) = V(f)Y + f\tilde{V}(Y) \). Any sheaf \( \mathcal{F} \) of \( \Theta \)-modules admitting such a lifting \( \tilde{V} \) is called \( \Theta \)-equivariant and \( \tilde{V} \) is called an equivariant lift of \( V \). Note that \( \tilde{V} \) defines \( V_0 \in H^0(X, \text{Hom}_\Theta(\Theta, \Theta) \otimes \mathcal{E})_2 \). Analogously, an equivariant lift \( \tilde{V}: \mathcal{F} \rightarrow \mathcal{F} \otimes L \) of \( V \in H^0(X, \mathcal{T} \otimes L) \) defines an element \( V_0 \in H^0(X, (\text{Hom}_\Theta(\mathcal{E}, \mathcal{E}) \otimes \mathcal{E})_2) \). Now \( \Theta \) is generally not equivariant for an arbitrary \( V \in H^0(X, \mathcal{T} \otimes L) \), however \( \tilde{V} \) admits a well-defined localization \( V_0 \in H^0(X, (\text{Hom}_\Theta(\Theta, \Theta) \otimes \mathcal{E})_2) \). In fact, choose a covering \( \{ U_a \} \) of \( X \) such that both \( L \mid U_a \) and \( T \mid U_a \) are locally trivial for each \( a \), and on \( U_a \) write \( V = w_a \otimes t_a \) where \( w_a \in H^0(U_a, \Theta) \), \( t_a \in H^0(U_a, \mathcal{E}) \), and \( t_a \) is nowhere vanishing. For \( Y \in H^0(U_a, \Theta) \), set \( \tilde{V}_a(Y) = [w_a, Y] \otimes t_a \). Note that on \( U_a \cap U_b \), \( \tilde{V}_b(Y) - \tilde{V}_a(Y) = t_b t_a^{-1} i(Y)d(t_b t_a^{-1}) \otimes V \), consequently the \( \tilde{V}_a \) patch on \( Z \) giving \( V_0 \) as asserted.

Let \( p(V_0) \) denote the element of \( H^0(X, \mathcal{E}_Z^n) \) obtained by applying the ad-invariant symmetric linear map \( p: \text{gl}(n, \mathbb{C}) \otimes k \rightarrow \mathbb{C} \) to \( V_0 \). In order to construct the class \( \phi_k(p(V_0)) \) in \( H^0(C_k) \), first choose a local holomorphic connection \( D_a \) for \( T \mid U_a \), and let \( D = \sum a D_a \) be the connection of type \( (1, 0) \) on \( T \) where \( a \) is a partition of unity fitted to \( \{ U_a \} \). Following [1], consider the \( \Theta \)-linear map \( \psi: \Omega^1 \rightarrow \text{Hom}(\Theta, \Theta) \otimes \Omega^1 \) such that if \( \omega \in \Omega^1, \) and \( Y \in \Theta \), then \( \psi(Y) \omega(Y) \) is the element of \( \text{Hom}(\Theta, \Theta) \) such that \( \psi(Y) \omega(Y) \) is \( \omega(w)Y \). Let \( \Gamma \in A^{1,0}(\text{Hom}(T, T)) \) be given by \( \Gamma = \sum a \{ \psi(t_a^{-1} dt_a) - D_a \} \), and let \( K^* = \tilde{\tau} \Gamma \in A^{1,1}(\text{Hom}(T, T)) \). Finally, define \( \tau = \sum a \{ V_a - i(V)D_a \} \in A^{0,0}(\text{Hom}(T, T) \otimes L) \). The following lemma is proved by a local calculation which will be omitted.

**Lemma 5.** \( i(V) K^* + \tilde{\tau} \tau = 0 \) in \( A^{0,1}(\text{Hom}(T, T) \otimes L) \).

One may therefore perform the ad-invariant map construction to get a class \( p(K^* + \tau) \) \( p((K^* + \tau) \otimes k) \) in \( C_k \) where \( k = \text{deg} p \). Because of Lemma 5, it follows that \( p(K^* + \tau) \) is a total cocycle. Note

\[ p(K^* + \tau) = p(K^* \otimes k) + kp(K^* \otimes k - 1 \otimes \tau) + \cdots + p(\tau \otimes k). \]

Note that by definition, \( p(\tau \otimes k) \) is an extension of \( p(V_0) \) to \( X \). If \( k = \text{deg} p = \)
we therefore get by (1) that
\[
\left(\frac{1}{2\pi i}\right)^n \int_X p(K^* \otimes \alpha) = \text{Res} p(V_0)
\]
which is part of the M.V.F. Theorem. The rest of the theorem is an identification of \((1/2\pi i)^n \int_X p(K^* \otimes \alpha)\) with a characteristic number of \(T - L^{-1}\).

In order to prove Theorem 2, note that in the case \(\deg p = k < n\), if \(\sigma \in H^0(X, L)\), then \(\sigma^{n-k} p(V_0)\) lies in \(H^0(X, L^k)\) and \(m_\phi_n(\sigma^{n-k} p(V_0)) = (-1)^m (\sigma^{n-k} \times p(K^{*k}))\) since \(\phi_n(\sigma^{n-k} p(V_0))\) is represented by the total cocycle \(\sigma^{n-k} p(K^* + \tau)\) of \(C^n_0\) (due to the fact that \(\sigma^{n-k}\) commutes with both \(i(V)\) and \(\partial\)). But clearly \(m(\sigma^{n-k} p(K^* + \tau)) = 0\) since \(\deg p < n\). Consequently \(\text{Res}(\sigma^{n-k} p(V_0)) = 0\), by (1), as asserted.

Remark. The notion of \(V\)-equivariance is studied in [1] and in [4] from a different viewpoint. In [4] it is shown that if \(X\) is projective, then given a holomorphic vector bundle \(E\) on \(X\), there exists a \(V \in H^0(X, T \otimes L)\) with isolated zeros for which \(E\) is \(V\) equivariant. Thus \(E\)'s characteristic numbers can be computed as above.

REFERENCES


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