SOME APPLICATIONS OF THE STONE-WEIERSTRASS
THEOREM

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ABSTRACT. Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $f \in A$. The Stone-Weierstrass theorem is used to obtain results on the function algebra on $X$ generated by the elements of $A$ and the function $f$.

Preliminaries. If $A$ is a collection of continuous functions on a compact Hausdorff space $X$ separating the points of $X$, $[A; X]$ denotes the function algebra on $X$ generated by the elements of $A$, i.e., the smallest algebra of continuous functions on $X$ containing the elements of $A$ and the constant functions which is uniformly closed in $C(X)$, i.e., closed in the algebra $C(X)$ of all continuous functions on $X$ provided with the supremum norm $\|f\|_X = \sup\{|f(x)|: x \in X\}$. If $f$ (respectively $A$) is a function (a collection of functions) on $X$, and $Y$ is a subset of $X$, then $f|Y (A|Y)$ denotes the restriction of $f$ to $Y$ (the collection of restrictions of elements of $A$ to $Y$).

If $A$ is a function algebra on $X$, a closed subset $K$ of $X$ is called a peak set for $A$ if there exists a function $f \in A$ such that $f(K) = \{1\}$ and $|f(x)| < 1$ for all $x \in X \setminus K$. If $K$ is a peak set for $A$ then $A|K$ is closed in $C(K)$ [3, p. 163].

A subset $K$ of $X$ is called a set of antisymmetry for $A$ [2, p. 60] if any $f \in A$ which is real-valued on $K$ is constant on $K$. The collection of maximal sets of antisymmetry is a closed pairwise disjoint cover of $X$. The generalized Stone-Weierstrass theorem reads:

If $f \in C(X)$ and $f|K \in A|K$ for all maximal sets of antisymmetry $K$ for $A$, then $f \in A$.

If $X$ is a compact subset of the complex plane, $P(X)$ is the function algebra on $X$ consisting of uniform limits on $X$ of polynomials; $R(X)$ is the function algebra on $X$ consisting of uniform limits on $X$ of rational functions with pole sets missing $X$.

The algebra $A(X)$ is the function algebra on $X$ consisting of all continuous functions on $X$ which are holomorphic on the interior of $X$.

Finally the complex homomorphism space of a function algebra $A$ is denoted by $\Delta A$. We abbreviate for $Y$ closed in $\Delta A : [A|Y] = [A|Y; Y]$.

In the following we determine the function algebra $[A, f; X]$ where $A$ is a
function algebra on \( X \) and \( f \) is a real-valued function on \( X \) or the complex conjugate of an element of \( A \). As corollaries we obtain results of Mergelyan, Minsker and Preskenis.

**Theorem 1.** Let \( A \) be a function algebra on \( X \), \( f \) a real-valued continuous function on \( X \). Let \( X_\alpha = \{ x \in X : f(x) = \alpha \}, \alpha \in f(X) \). Then \([A, f; X] = \{ g \in C(X) : g|X_\alpha \in [A|X_\alpha] \text{ for each } \alpha \in f(X) \} \).

**Proof.** The inclusion \( \subset \) is trivial. Conversely, let \( K \) be a maximal set of antisymmetry for \([A, f; X]\), then \( K \subset X_{\alpha_0} \) for some \( \alpha_0 \). Any \( g \in C(X) \) such that \( g|X_\alpha \in [A|X_\alpha] \) for each \( \alpha \) has the property that \( g|X_{\alpha_0} \in [A|X_{\alpha_0}] \subset ([A, f; X]|X_{\alpha_0}) = [A, f; X]|X_{\alpha_0} \). The last equality follows from the fact that \( X_{\alpha_0} \) is a peak set for \([A, f; X]\). So also \( g|K \in [A, f; X]|K \), so by the Stone-Weierstrass theorem \( g \in [A, f; X] \).

An immediate consequence is the following result of Mergelyan [4].

**Theorem 2 (Mergelyan).** Let \( X \) be a compact subset of \( C \); \( f \) a real-valued continuous function on \( X \) such that \( X_\alpha = \{ x \in X : f(x) = \alpha \} \) is polynomially convex for each \( \alpha \in f(X) \). Then \([z, f; X] = [R(X), f; X]\) and this function algebra consists of all elements of \( C(X) \) which are holomorphic on the interior of the sets \( X_\alpha \).

**Proof.** Let \( A = P(X) \) and apply Theorem 1. By the classical Mergelyan theorem \([A|X_\alpha] = P(X_\alpha) = A(X_\alpha) \). Trivially \([z, f; X] \subset [R(X), f; X] \subset \{ g \in C(X) : g \text{ is holomorphic on the interior of the sets } X_\alpha \} \).

**Theorem 3.** Let \( A \) be a function algebra on \( X \), \( f \in A \). Then \([A, \tilde{f}; X] = \{ g \in C(X) : g|X_\alpha \in [A|X_\alpha] \text{ for each } \alpha \in f(X) \} \), where \( X_\alpha = \{ x \in X : f(x) = \alpha \} \).

**Proof.** Again the inclusion \( \subset \) is trivial. Since both \( \text{Re} f \) and \( \text{Im} f \) belong to \([A, \tilde{f}; X]\), \( X_\alpha \) is the intersection of the peak sets for \([A, \tilde{f}; X] : \{ x \in X : \text{Re} f(x) = \text{Re} \alpha \}, \{ x \in X : \text{Im} f(x) = \text{Im} \alpha \}, \) so is itself a peak set for \([A, \tilde{f}; X]\). Also if \( K \) is a maximal set of antisymmetry for \([A, \tilde{f}; X]\), \( K \) is contained in some \( X_\alpha \) (since both \( \text{Re} f \) and \( \text{Im} f \) are constant on \( K \)). The conclusion now follows as in the proof of Theorem 1.

Applying this result to the algebra \( R(X) \), \( X \subset C \), we obtain

**Theorem 4.** Let \( X \) be a compact subset of the complex plane.

(i) Let \( g \in R(X) \) and let \( X_\alpha = \{ x \in X : g(x) = \alpha \}, \alpha \in g(X) \). Then \([R(X), \tilde{g}; X] = \{ f \in C(X) : f|X_\alpha \in R(X_\alpha) \text{ for each } \alpha \in g(X) \} \).

(ii) Let \( g \in R(X) \) such that the level sets \( X_\alpha \) of \( g \) are polynomially convex. Then \([R(X), \tilde{g}; X]\) consists of all elements of \( C(X) \) which are holomorphic on the interior of the sets \( X_\alpha \).

In particular:

(iii) If \( X \) is polynomially convex and \( g \in P(X) \) then \([z, \tilde{g}; X]\) consists of all elements of \( C(X) \) which are holomorphic on the interior of the level sets of \( g \).

**Proof.** For a compact subset \( Y \) of \( X \), \([R(X)|Y] = R(Y) \) if and only if \( Y \) is
\[ R(X) \text{-convex. Since } X_\alpha \text{ is } R(X) \text{-convex } \{ R(X) | X_\alpha \} = R(X_\alpha). \text{ Now apply Theorem } 3 \text{ and } (i) \text{ follows. Using the classical Mergelyan theorem } (ii) \text{ and } (iii) \text{ follow immediately.}

**Theorem 5.** Let \( X \) be a compact set of the complex plane.

(i) Let \( g \in R(X) \) such that the level sets \( X_\alpha \) of \( g \) are polynomially convex and such that \( g \) is not constant on any of the components of the interior of \( X \). Then 
\[
[R(X), \bar{g}; X] = C(X).
\]

In particular (Preskenis [6]):

(ii) If \( X \) is polynomially convex and \( g \in P(X) \) such that \( g \) is not constant on any of the components of the interior of \( X \) then 
\[
[z, \bar{g}; X] = C(X).
\]

This result follows immediately from Theorem 4 since the hypothesis on \( g \) implies that the interior of the sets \( X_\alpha \) are empty. An abstract version of Theorem 5(ii) is

**Theorem 6.** Let \( A \) be a function algebra on \( X \). Let \( f \in A \) and let \( Y \) be the polynomially convex hull of \( f(X) \). Let \( g \in P(Y) \) such that \( g \) is not constant on any of the components of the interior of \( Y \). Then 
\[
[A, f; X] = A, g \circ f; X].
\]

**Proof.** Since \( g \in C(Y), \bar{g} \) is uniformly approximable on \( Y \) by polynomials in \( z \) and \( \bar{z} \), so the inclusion "\( \supseteq \)" follows since \( g \circ f \in \{ A, f; X \]. Conversely, by Theorem 5(ii) it follows that 
\[
[z, \bar{g}, Y] = C(Y), \text{ hence } \bar{z} \in [z, \bar{g}; Y] \text{ so } f \in [A, g \circ f; X].
\]

Applying Theorem 5(ii) we obtain a result of Minsker [5]:

**Corollary 1 (Minsker).** Let \( X \) be a compact subset of the complex plane and \( m \in \mathbb{N} \). Then 
\[
[z, z^m; X] = C(X).
\]

**Corollary 2.** Let \( f \) and \( g \) be holomorphic on a neighborhood of \( 0 \in \mathbb{C} \) such that \( \partial f / \partial z(0) \neq 0 \) and \( g \) is not constant near \( 0 \). Then there exists a closed disc \( D \) centered at the origin such that 
\[
[f, \bar{g}; D] = [g, f; D] = C(D).
\]

**Proof.** Without loss of generality \( f(0) = 0 \), so if \( \delta > 0 \) is small enough \( z \) can be approximated uniformly on \( D = \{|z| < \delta\} \) by polynomials in \( f \). Moreover we may assume that \( g \) is defined on \( D \). By Theorem 5(ii) 
\[
[z, \bar{g}; D] = C(D), \text{ so } [f, \bar{g}; D] = C(D). \text{ And } [g, f; D] = [g, \bar{z}; D] \text{ consists of the complex conjugates of the elements of } [\bar{g}, z; D]. \text{ So } [g, \bar{f}; D] = C(D).
\]

**Corollary 3.** Let \( X = \{1 < |z| < 2\} \subset \mathbb{C} \text{ and let } r_1, r_2 \in R(X) \text{ such that } r_1 \notin P(X) \text{ and } r_2 \text{ is not constant on } X. \text{ Then } [z, r_1, r_2; X] = C(X).

**Proof.** Since \( P(X) \) is maximal in \( R(X) \) \([1], [z, r_1, X] = R(X) \) and by Theorem 5(i) 
\[
[R(X), \bar{r}_2; X] = C(X).
\]

Using techniques similar to those in the previous section we prove two other results.

**Theorem 7.** Let \( n, m \in \mathbb{N} \) such that \( \gcd(n, m) = 1 \) and let \( D = \{|z| < 1\} \). If \( f \) is a nonconstant element of \( P(D) \), then 
\[
[z^n, z^m, f; D] = C(D).
\]
Proof. There exists \( N \in \mathbb{N} \) such that \( z^k \in A = [z^n, z^m; D] \) for each \( k \geq N \). Without loss of generality we may assume \( f(0) = 0 \), so \( f^N \) is uniformly approximable on \( D \) by elements of \( A \).

Therefore \( \text{Re} f^N \) and \( \text{Im} f^N \) belong to \([A, f; D]\). Let \( K \) be a maximal set of antisymmetry for \([A, f; D]\). Then \( K \) is contained in a level set \( L \) of \( f^N \) which is a peak set for \([A, f; D]\). Since \( f^N \) is not constant on \( D \), \( L = P \cup \{a_1, a_2, a_3, \ldots \} \) where \( P \) is a proper subset of \( \{|z| = 1\} \) and where \( \{a_1, a_2, a_3, \ldots\} \) is a discrete subset of \( \{|z| < 1\} \). Now \( \Delta A = D \). (Indeed: let \( \phi \in \Delta A \) such that \( \phi(z^N) = 0 \). If \( k \in \mathbb{N} \) such that \( z^k \in A \) then \( \phi(z^k)^N \phi(z^N) = 0 \), so \( \phi(z^k) = 0 \) hence \( \phi \) is point evaluation at the point \( 0 \). If \( \phi(z^N) \neq 0 \), let \( a = \phi(z^{N+1})/\phi(z^N) \). For \( k \in \mathbb{N} \) such that \( z^k \in A \) we have \( \phi(z^k) \cdot \phi(z^{Nk}) = \phi(z^{(N+1)k}) \) so \( \phi(z^k) = a^k \), hence \( \phi \) is point evaluation at \( a \in D \).)

Since \( a_n \) is an isolated point of the peak set \( L \), by Rossi's local peak set theorem [2, p. 91] there exists \( f_n \in [A, f; D] \) such that \( f_n(x) = 1 \) for each \( x \in L \setminus \{a_n\} \) and such that \( |f_n(x)| < 1 \) for each \( x \in \{a_n\} \cup D \setminus L \). Since \([A, f; D]|L \) is closed we may assume \( f_n(a_n) = 0 \) [3, p. 164]. So the maximal set \( K \) of antisymmetry reduces to a single point \( \{a_n\} \) or else \( K \subset P \).

Now \( P = L \cap \cap_{n=1}^{\infty} \{x \in D : f_n(x) = 1\} \) is again a peak set for \([A, f; D]\) and is convex relative to the algebra \( A \) (since \( L \) is). So \( \Delta[A, f; D]|P] = \Delta[A|P] = P \). Since \( z^n \) and \( z^m \) have no zeros on \( P \), \( z^{-n}, z^{-m} \in [[A, f; D]|P] \), so \([A, f; D]|P] = C(P) \) and since \([A, f; D]|P] \) is closed in \( C(P) \) we have \([A, f; D]|K = C(K) \). By the Stone-Weierstrass theorem it follows that \([A, f; D] = C(D) \).

Theorem 8 (Minsker [5]). Let \( n, m \in \mathbb{N} \) such that \( \gcd(n, m) = 1 \) and let \( X \) be a compact subset of \( C \). Then \([z^n, z^m; X]\) = \( C(X) \).

Proof. \( \text{Re} z^{nm} \) and \( \text{Im} z^{nm} \) belong to \( A = [z^n, z^m; X] \), so a maximal set of antisymmetry for \( A \) consists of a finite number of points, so has to be a singleton. By the Stone-Weierstrass theorem \( A = C(X) \).

We conclude with two questions. Let \( D = \{|z| < 1\} \) and suppose that \( f \in P(D) \) and \( z^2 \) separate the points of \( D \). What can be said about the algebra \([z^2, f; D]\)?

Let \( X = \{1 \leq |z| < 2\} \). If \( f \in R(X) \) and \( z^2 \) separate points of \( X \), determine \([z^2, z^{-2}, f; X]\).

References


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