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SOME APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM

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ABSTRACT. Let A be a function algebra on a compact Hausdorff space X and let $f \in A$. The Stone-Weierstrass theorem is used to obtain results on the function algebra on X generated by the elements of A and the function $\bar{f}$.

Preliminaries. If A is a collection of continuous functions on a compact Hausdorff space X separating the points of X, $[A; X]$ denotes the function algebra on X generated by the elements of A, i.e., the smallest algebra of continuous functions on X containing the elements of A and the constant functions which is uniformly closed in $C(X)$, i.e., closed in the algebra $C(X)$ of all continuous functions on X provided with the supremum norm $\|\cdot\|_X : \|f\|_X = \sup\{|f(x)|: x \in X\}$. If f (respectively A) is a function (a collection of functions) on X, and Y is a subset of X, then $f|Y$ $(A|Y)$ denotes the restriction of f to Y (the collection of restrictions of elements of A to Y).

If A is a function algebra on X, a closed subset K of X is called a peak set for A if there exists a function $f \in A$ such that $f(K) = \{1\}$ and $|f(x)| < 1$ for all $x \in X \setminus K$. If K is a peak set for A then $A \mid K$ is closed in $C(K)$ [3, p. 163].

A subset K of X is called a set of antisymmetry for A [2, p. 60] if any $f \in A$ which is real-valued on K is constant on K. The collection of maximal sets of antisymmetry is a closed pairwise disjoint cover of X. The generalized Stone-Weierstrass theorem reads:

> If $f \in C(X)$ and $f \mid K \in A \mid K$ for all maximal sets of antisymmetry K for A, then $f \in A$.

If X is a compact subset of the complex plane, $P(X)$ is the function algebra on X consisting of uniform limits on X of polynomials; $R(X)$ is the function algebra on X consisting of uniform limits on X of rational functions with pole sets missing X.

The algebra $A(X)$ is the function algebra on X consisting of all continuous functions on X which are holomorphic on the interior of X.

Finally the complex homomorphism space of a function algebra A is denoted by ΔA. We abbreviate for Y closed in ΔA : $[A|Y] = [A|Y; Y]$.

In the following we determine the function algebra $[A, f; X]$ where A is a

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function algebra on X and f is a real-valued function on X or the complex conjugate of an element of A. As corollaries we obtain results of Mergelyan, Minsker and Preskenis.

THEOREM 1. *Let A be a function algebra on X, f a real-valued continuous function on X. Let $X_\alpha = \{x \in X : f(x) = \alpha\}$, $\alpha \in f(X)$. Then $[A, f; X] = \{g \in C(X) : g|X_\alpha \in [A|X_\alpha]$ for each $\alpha \in f(X)\}$.*

PROOF. The inclusion "$\subset$" is trivial. Conversely, let K be a maximal set of antisymmetry for $[A, f; X]$, then $K \subset X_{\alpha_0}$ for some α_0. Any $g \in C(X)$ such that $g|X_\alpha \in [A|X_\alpha]$ for each α has the property that $g|X_{\alpha_0} \in [A|X_{\alpha_0}] \subset [[A, f; X]|X_{\alpha_0}] = [A, f; X]|X_{\alpha_0}$. The last equality follows from the fact that X_{α_0} is a peak set for $[A, f; X]$. So also $g|K \in [A, f; X]|K$, so by the Stone-Weierstrass theorem $g \in [A, f; X]$.

An immediate consequence is the following result of Mergelyan [4].

THEOREM 2 (MERGELYAN). *Let X be a compact subset of $\mathbf{C}$; f a real-valued continuous function on X such that $X_\alpha = \{x \in X : f(x) = \alpha\}$ is polynomially convex for each $\alpha \in f(X)$. Then $[z, f; X] = [R(X), f; X]$ and this function algebra consists of all elements of $C(X)$ which are holomorphic on the interior of the sets X_α.*

PROOF. Let $A = P(X)$ and apply Theorem 1. By the classical Mergelyan theorem $[A|X_\alpha] = P(X_\alpha) = A(X_\alpha)$. Trivially $[z, f; X] \subset [R(X), f; X] \subset \{g \in C(X) : g$ is holomorphic on the interior of the sets $X_\alpha\}$.

THEOREM 3. *Let A be a function algebra on X, $f \in A$. Then $[A, \bar{f}; X] = \{g \in C(X) : g|X_\alpha \in [A|X_\alpha]$ for each $\alpha \in f(X)\}$, where $X_\alpha = \{x \in X : f(x) = \alpha\}$, $\alpha \in f(X)$.*

PROOF. Again the inclusion "$\subset$" is trivial. Since both $\operatorname{Re} f$ and $\operatorname{Im} f$ belong to $[A, \bar{f}; X]$, X_α is the intersection of the peak sets for $[A, \bar{f}; X] : \{x \in X : \operatorname{Re} f(x) = \operatorname{Re} \alpha\}$, $\{x \in X : \operatorname{Im} f(x) = \operatorname{Im} \alpha\}$, so is itself a peak set for $[A, \bar{f}; X]$. Also if K is a maximal set of antisymmetry for $[A, \bar{f}; X]$, K is contained in some X_α (since both $\operatorname{Re} f$ and $\operatorname{Im} f$ are constant on K). The conclusion now follows as in the proof of Theorem 1.

Applying this result to the algebra $R(X)$, $X \subset \mathbf{C}$, we obtain

THEOREM 4. *Let X be a compact subset of the complex plane.*

(i) *Let $g \in R(X)$ and let $X_\alpha = \{x \in X : g(x) = \alpha\}$, $\alpha \in g(X)$. Then $[R(X), \bar{g}; X] = \{f \in C(X) : f|X_\alpha \in R(X_\alpha)$ for each $\alpha \in g(X)\}$.*

(ii) *Let $g \in R(X)$ such that the level sets X_α of g are polynomially convex. Then $[R(X), \bar{g}; X]$ consists of all elements of $C(X)$ which are holomorphic on the interior of the sets X_α.*

In particular:

(iii) *If X is polynomially convex and $g \in P(X)$ then $[z, \bar{g}; X]$ consists of all elements of $C(X)$ which are holomorphic on the interior of the level sets of g.*

PROOF. For a compact subset Y of X, $[R(X)|Y] = R(Y)$ if and only if Y is

$R(X)$-convex. Since X_α is $R(X)$-convex $[R(X)|X_\alpha] = R(X_\alpha)$. Now apply Theorem 3 and (i) follows. Using the classical Mergelyan theorem (ii) and (iii) follow immediately.

THEOREM 5. *Let X be a compact set of the complex plane.*

(i) *Let $g \in R(X)$ such that the level sets X_α of g are polynomially convex and such that g is not constant on any of the components of the interior of X. Then* $[R(X), \bar{g}; X] = C(X)$.

In particular (*Preskenis* [**6**]):

(ii) *If X is polynomially convex and $g \in P(X)$ such that g is not constant on any of the components of the interior of X then* $[z, \bar{g}; X] = C(X)$.

This result follows immediately from Theorem 4 since the hypothesis on g implies that the interior of the sets X_α are empty. An abstract version of Theorem 5(ii) is

THEOREM 6. *Let A be a function algebra on X. Let $f \in A$ and let Y be the polynomially convex hull of $f(X)$. Let $g \in P(Y)$ such that g is not constant on any of the components of the interior of Y. Then* $[A, \bar{f}; X] = [A, \bar{g} \circ f; X]$.

PROOF. Since $g \in C(Y)$, $\bar{g}$ is uniformly approximable on Y by polynomials in z and $\bar{z}$, so the inclusion "$\supset$" follows since $\bar{g} \circ f \in [A, \bar{f}; X]$. Conversely, by Theorem 5(ii) it follows that $[z, \bar{g}; Y] = C(Y)$, hence $\bar{z} \in [z, \bar{g}; Y]$ so $\bar{f} \in [A, \bar{g} \circ f; X]$.

Applying Theorem 5(ii) we obtain a result of Minsker [**5**]:

COROLLARY 1 (MINSKER). *Let X be a compact subset of the complex plane and $m \in \mathbf{N}$. Then* $[z, \bar{z}^m; X] = C(X)$.

COROLLARY 2. *Let f and g be holomorphic on a neighborhood of $0 \in \mathbf{C}$ such that $\partial f/\partial z(0) \neq 0$ and g is not constant near 0. Then there exists a closed disc D centered at the origin such that* $[f, \bar{g}; D] = [g, \bar{f}; D] = C(D)$.

PROOF. Without loss of generality $f(0) = 0$, so if $\delta > 0$ is small enough z can be approximated uniformly on $D = \{|z| \leqslant \delta\}$ by polynomials in f. Moreover we may assume that g is defined on D. By Theorem 5(ii) $[z, \bar{g}; D] = C(D)$, so $[f, \bar{g}; D] = C(D)$. And $[g, \bar{f}; D] = [g, \bar{z}; D]$ consists of the complex conjugates of the elements of $[\bar{g}, z; D]$. So $[g, \bar{f}; D] = C(D)$.

COROLLARY 3. *Let $X = \{1 \leqslant |z| \leqslant 2\} \subset \mathbf{C}$ and let $r_1, r_2 \in R(X)$ such that $r_1 \notin P(X)$ and r_2 is not constant on X. Then* $[z, r_1, \bar{r}_2; X] = C(X)$.

PROOF. Since $P(X)$ is maximal in $R(X)$ [**1**], $[z, r_1; X] = R(X)$ and by Theorem 5(i) $[R(X), \bar{r}_2; X] = C(X)$.

Using techniques similar to those in the previous section we prove two other results.

THEOREM 7. *Let $n, m \in \mathbf{N}$ such that* $\gcd(n, m) = 1$ *and let $D = \{|z| \leqslant 1\}$. If f is a nonconstant element of $P(D)$, then* $[z^n, z^m, \bar{f}; D] = C(D)$.

PROOF. There exists $N \in \mathbf{N}$ such that $z^k \in A = [z^n, z^m; D]$ for each $k \geqslant N$. Without loss of generality we may assume $f(0) = 0$, so f^N is uniformly approximable on D by elements of A.

Therefore $\operatorname{Re} f^N$ and $\operatorname{Im} f^N$ belong to $[A, \bar{f}; D]$. Let K be a maximal set of antisymmetry for $[A, \bar{f}; D]$. Then K is contained in a level set L of f^N which is a peak set for $[A, \bar{f}; D]$. Since f^N is not constant on D, $L = P \cup \{a_1, a_2, a_3, \ldots\}$ where P is a proper subset of $\{|z| = 1\}$ and where $\{a_1, a_2, a_3, \ldots\}$ is a discrete subset of $\{|z| < 1\}$. Now $\Delta A = D$. (Indeed: let $\phi \in \Delta A$ such that $\phi(z^N) = 0$. If $k \in \mathbf{N}$ such that $z^k \in A$ then $(\phi(z^k))^N(\phi(z^N))^k = 0$, so $\phi(z^k) = 0$ hence ϕ is point evaluation at the point 0. If $\phi(z^N) \neq 0$, let $a = \phi(z^{N+1})/\phi(z^N)$. For $k \in \mathbf{N}$ such that $z^k \in A$ we have $\phi(z^k) \cdot \phi(z^{Nk}) = \phi(z^{(N+1)k})$ so $\phi(z^k) = a^k$, hence ϕ is point evaluation at $a \in D$.)

Since a_n is an isolated point of the peak set L, by Rossi's local peak set theorem [2, p. 91] there exists $f_n \in [A, \bar{f}; D]$ such that $f_n(x) = 1$ for each $x \in L \setminus \{a_n\}$ and such that $|f_n(x)| < 1$ for each $x \in \{a_n\} \cup D \setminus L$. Since $[A, \bar{f}; D]|L$ is closed we may assume $f_n(a_n) = 0$ [3, p. 164]. So the maximal set K of antisymmetry reduces to a single point $\{a_n\}$ or else $K \subset P$.

Now $P = L \cap \bigcap_{n=1}^{\infty}\{x \in D : f_n(x) = 1\}$ is again a peak set for $[A, \bar{f}; D]$ and is convex relative to the algebra A (since L is). So $\Delta[[A, \bar{f}; D]|P] = \Delta[A|P] = P$. Since z^n and z^m have no zeros on P, $z^{-n}, z^{-m} \in [[A, \bar{f}; D]|P]$, so $[[A, \bar{f}; D]|P] = C(P)$ and since $[A, \bar{f}; D]|P$ is closed in $C(P)$ we have $[A, \bar{f}; D]|P = C(P)$. So $[A, \bar{f}; D]|K = C(K)$. By the Stone-Weierstrass theorem it follows that $[A, \bar{f}; D] = C(D)$.

THEOREM 8 (MINSKER [5]). *Let $n, m \in N$ such that* $\gcd(n, m) = 1$ *and let X be a compact subset of* $\mathbf{C}$. *Then* $[z^n, \bar{z}^m; X] = C(X)$.

PROOF. $\operatorname{Re} z^{nm}$ and $\operatorname{Im} z^{nm}$ belong to $A = [z^n, \bar{z}^m; X]$, so a maximal set of antisymmetry for A consists of a finite number of points, so has to be a singleton. By the Stone-Weierstrass theorem $A = C(X)$.

We conclude with two questions. Let $D = \{|z| \leqslant 1\}$ and suppose that $f \in P(D)$ and z^2 separate the points of D. What can be said about the algebra $[z^2, \bar{f}; D]$?

Let $X = \{1 \leqslant |z| \leqslant 2\}$. If $f \in R(X)$ and z^2 separate points of X, determine $[z^2, z^{-2}, \bar{f}; X]$.

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