METRIC CHARACTERIZATIONS OF DIMENSION
FOR SEPARABLE METRIC SPACES

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Abstract. A subset $B$ of a metric space $(X, d)$ is called a $d$-bisector set iff there are distinct points $x$ and $y$ in $X$ with $B = \{z : d(x, z) = d(y, z)\}$. It is shown that if $X$ is a separable metrizable space, then $\dim(X) < n$ iff $X$ has an admissible metric $d$ for which $\dim(B) < n - 1$ whenever $B$ is a $d$-bisector set. For separable metrizable spaces, another characterization of $n$-dimensionality is given as well as a metric dependent characterization of zero dimensionality.

1. Introduction. Let $d$ be a metric for a set $X$. A subset $B$ of $X$ is called a $d$-bisector set in $X$ iff there are distinct points $x_1$ and $x_2$ in $X$ such that $B = \{x : d(x, x_1) = d(x, x_2)\}$. In this note we will show that if $X$ is a separable metrizable space, then $\dim(X) < n$ iff $X$ has an admissible totally bounded metric $d$ such that if $B$ is any $d$-bisector set in $X$, then $\dim(B) < n - 1$. Using this result and the machinery of [4], we observe that $t(X) = \dim(X)$ for every separable metrizable space $X$, where $t(X)$ is defined in the same way as in the first author's definition of the reduced bisector dimension function $r(X)$ except that only totally bounded metrics are considered.

A metric $d$ for a set $X$ is said to be strongly rigid provided that if the two element subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ of $X$ are distinct, then $d(x_1, x_2) \neq d(y_1, y_2)$. With respect to strongly rigid metrics, see [1], [2], [5] and [6]. We define a metric $d$ on a set $X$ to be star rigid iff whenever $x, y$ and $z$ are points of $X$ with $y \neq z$, then $d(x, y) \neq d(x, z)$. Since every strongly rigid metric is star rigid, we have that if $X$ is separable metrizable and $\dim(X) = 0$, then $X$ has an admissible totally bounded star rigid metric [1]. We shall show below that if a nonempty space $X$ has an admissible totally bounded star rigid metric, then $\dim(X) = 0$.

We shall use [7] as a general reference on dimension theory.

2. Theorems. Combining Lemma 3.1 and Theorem 1.2 of [4], one obtains the following: if $X$ is a compact metrizable space with $\dim(X) = n$, then $X$ has an admissible metric $d$ such that if $B$ is a $d$-bisector set in $X$, then $\dim(B) < n - 1$. Using this result, we establish the following more general theorem.

Theorem 1. Let $X$ be a separable metrizable space. Then, $\dim(X) < n$ iff $X$
has an admissible totally bounded metric $d$ such that if $B$ is a $d$-bisector set in $X$, then $\dim(B) \leq n - 1$.

**Proof.** Let $\dim(X) < n$. The space $X$ may be embedded in a compact metrizable space $Y$ where $\dim(Y) < n$. Choose an admissible metric $\rho$ for $Y$ such that if $B$ is a $\rho$-bisector set in $Y$, then $\dim(B) < n - 1$. Let $d$ be the metric for $X$ obtained by restricting $\rho$ to $X \times X$. Since $\rho$ is a metric for a compact space, $\rho$ is totally bounded and $d$, being a restriction of $\rho$, is also a totally bounded metric. If $A \subset X$ is a $d$-bisector set in $X$, then $A = \{x: d(x, x_1) = d(x, x_2)\}$ for a pair of points $x_1$ and $x_2$ in $X$. Then $A = B \cap X$ where $B = \{z \in Y: \rho(z, x_1) = \rho(z, x_2)\}$ and since $\dim(B) < n - 1$, necessarily $\dim(A) < n - 1$.

Conversely, let $d$ be a totally bounded metric for $X$ such that $\dim(B) < n - 1$ for every $d$-bisector set $B \subset X$. Let $x \in X$ and $e > 0$ be arbitrarily chosen. Our aim is to show that there is a base at $x$ whose boundaries have dimension $< n - 1$. In case $x$ is an isolated point the statement is trivial. In the opposite case there exist points $x_1, x_2, \ldots, x_m$, none equal to $x$, such that the $e$-balls $S_e(x_i)$ centered on them cover $X$. Let $H_i = \{z: d(x, z) < d(z, x_i)\}$ for $i = 1, 2, \ldots, m$. Note that

$$\text{Bd}(H_i) \subset \{z: d(x, z) = d(z, x_i)\}$$

so that $\dim(\text{Bd}(H_i)) < n - 1$. Also note that $H_i$ is open for each $i$ so that $H = \cap \{H_i: i = 1, \ldots, m\}$ is an open neighborhood of $x$. Since $\text{Bd}(H) \subset \cup \{\text{Bd}(H_i)\}$, we have that $\dim(\text{Bd}(H)) < n - 1$. Let $z \in X - S_e(x)$. There exists an $x_i$ with $d(z, x_i) < e$; since $d(z, x) > e$, necessarily $z \not\in H_i$. But then $z \not\in H$, that is, $H \subset S_e(x)$. It follows that the point $x$ has a neighborhood base consisting of open sets having boundaries of dimension less than $n$ from which it follows that $\dim(X) < n$, completing the proof.

Using Theorem 1 and the principal result of [1] we now establish the following theorem.

**Theorem 2.** Let $X$ be a separable metrizable space. Then, $\dim(X) = 0$ iff $X$ is nonempty and has an admissible totally bounded star rigid metric.

**Proof.** Let $X$ be a separable metrizable space with $\dim(X) = 0$. The space $X$ is homeomorphic to a subspace $S$ of the Cantor set $C$. But in [1] it was shown that the space $C$ has an admissible strongly rigid metric $\rho$ and the restriction of $\rho$ to $S \times S$ yields an admissible totally bounded strongly rigid (hence star rigid) metric for $S$, hence for $X$.

Conversely, let $d$ be an admissible totally bounded star rigid metric for the nonempty space $X$. If $x_1$ and $x_2$ are distinct points of $X$, then $\{z: d(x_1, z) = d(x_2, z)\} = \emptyset$ because $d$ is star rigid, that is, if $B$ is any $d$-bisector set in $X$, then $\dim(B) = -1$. By Theorem 1 it follows that $\dim(X) < 0$ and since $X$ is nonempty, $\dim(X) = 0$, completing the proof.
Let \((X, d)\) be a metric space. We write \(Y \supset Z\) iff \(Z \subset Y \subset X\) and \(Z\) is a \(\rho\)-bisector set in \(Y\) where \(\rho\) denotes the restriction of \(d\) to \(Y \times Y\). A reduced chain of length \(n\) is a chain

\[
x = X_0 \supset X_1 \supset \cdots \supset X_{n-1} \supset X_n
\]
such that \(\dim(X_n) < 0\) and \(\dim(X_{n-1}) > 0\). Let \(r(X, d) = n\) if there exists a reduced chain of length \(n\) but no reduced chain of length greater than \(n\). If there exist reduced chains of arbitrarily great length, let \(r(X, d) = \infty\). Define \(r(X)\) to be the minimum of \(r(X, d)\) taken over the set of all metrizations of the space \(X\). The dimension function \(r(X)\) was introduced in [4] where it was shown that \(r(X) = \dim(X)\) for every compact metrizable space \(X\). We modify \(r(X)\) for separable metrizable spaces \(X\) by letting \(r(X)\) be the minimum of \(r(X, d)\) taken over the set of all admissible totally bounded metrics for the space \(X\). Using Theorem 1 of this note in place of Theorem 1.1 of [4], the proof in [4] that \(r(X) = \dim(X)\) for compact metrizable \(X\) carries over to establish the following:

**Theorem 3.** If \(X\) is a separable metrizable space, then \(t(X) = \dim(X)\).

3. Questions. In [6] it was shown that if \(X\) is a metrizable space with \(\dim(X) = 0\) and \(\text{card}(X) < c\), the cardinality of the continuum, then \(X\) has an admissible strongly rigid metric. The question of whether this result remains true if we replace the condition that \(\dim(X) = 0\) by the hypothesis that \(\text{ind}(X) = 0\) appears difficult and to our knowledge has not been answered. The following related question may be more tractable.

**Question 1.** If \(X\) is a metrizable space with \(\text{ind}(X) = 0\) and \(\text{card}(X) < c\), must the space \(X\) have an admissible star rigid metric?

In [1] it was shown that if a metrizable space \(X\) has an admissible strongly rigid metric, then \(\text{ind}(X) = 0\). This fact together with Theorem 2 suggests the following question.

**Question 2.** If a metrizable space \(X\) has an admissible star rigid metric, must \(\text{ind}(X) = 0\) ?

The dimension function \(r\) is a modification of the dimension function \(b\) of [3] and, from the definitions, it is clear that \(r(X) \leq b(X)\) for any metrizable space \(X\). We also have \(r(X) \leq t(X)\) for any separable metrizable space \(X\).

**Question 3.** Do any of the dimension numbers \(r(X), b(X)\) and \(t(X)\) coincide on the class of all separable metrizable spaces?

**References**


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