AVOIDING SELF-REFERENTIAL STATEMENTS

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Abstract. Recursion-theoretic proofs of metamathematical results tend to rely on a pair of effectively inseparable r.e. sets and its properties. We establish a special property for a small configuration of such pairs and derive from it some metamathematical results not previously accessible to recursion-theoretic techniques.

0. Introduction. The applications of the dual completeness of a pair of effectively inseparable r.e. sets to metamathematical questions are manifold. Since Shepherdson 1960, however, more powerful results have been obtainable by diagonalization within a given theory. In this note, we prove a generalization of Smullyan’s dual completeness result (cf. Rogers 1967, Exercise 11.29) and list some metamathematical corollaries not previously obtainable recursion-theoretically.

We let \([e]\) denote the partial recursive function with index \(e\), and \(W_e\) the r.e. set with index \(e\). \(Txy\) is Kleene’s \(T\)-predicate and, for any assertions, \(\exists vRv\), \(\exists vSv\), with \(R, S\) recursive, we write

\[
\exists vRv \leq \exists vSv: \exists v[Rv \land \forall v' < v \exists v'' Sv'],
\]

\[
\exists vRv < \exists vSv: \exists v[Rv \land \forall v' < v \exists v'' \neg Sv'].
\]

A disjunction \(\exists vTv \lor \exists vUv\) in one of these contexts is assumed rewritten \(\exists v(Tv \lor Uv)\). For r.e. sets \(X, Y\), we define

\[
X \leq Y: \{x: x \in X \leq x \in Y\}, \quad X < Y: \{x: x \in X < x \in Y\},
\]

where \(x \in X, x \in Y\) abbreviate \(\exists vTexv\) for appropriate \(e\). Note that \(X \leq Y\) and \(Y < X\) are simply the sets obtained by applying the Reduction Theorem to \(X, Y\). (This notation is due to Dave Guaspari.)

1. A double dual completeness theorem. The main result of this note is the following

Theorem. Let \((A, C), (B, D)\) be pairs of effectively inseparable r.e. sets with \(A \subseteq B, C \subseteq D\). There is a recursive function \(f\) such that, for all \(x,\)

\[
x \in A \iff fx \in A \iff fx \in B;
\]

\[
x \in C \iff fx \in C \iff fx \in D.
\]

In words, the conclusion of the theorem simply states that the pair \((A, C)\) is uniformly many-one reducible to both pairs \((A, C)\) and \((B, D)\).

Proof. The proof is simple but devious. By Smullyan’s dual completeness...
result, there is a recursive function \( g \) such that, for all \( i, j \), the function \( [g(i, j)] \) reduces the pair \((W_i \leq W_j, W_j < W_i)\) to \((A, C)\). Apply Smullyan's Double Recursion Theorem (Rogers 1967, Theorem 11.10) to obtain indices \( a, c \) such that, for \( f = [g(a, c)] \) and all \( x \),

\[
x \in W_a \iff [fx \in D \lor x \in A] \land [fx \in B \lor x \in C],
\]

\[
x \in W_c \iff [fx \in B \lor x \in C] \land [fx \in D \lor x \in A].
\]

Obviously, \( W_a \) and \( W_c \) are disjoint.

Claim 1. \( W_a = A \leq C = A \); \( W_c = C < A = C \).

To see this, observe

\[
x \in W_a \Rightarrow x \in W_a - W_c
\]

\[
\Rightarrow fx \in A \subseteq B \land fx \notin D, \text{ since } A \cap D = \emptyset
\]

\[
\Rightarrow x \in A, fx \in B \lor x \in C, \text{ by definition of } W_a
\]

\[
\Rightarrow x \in A.
\]

Similarly, \( x \in W_c \Rightarrow x \in C \). But also,

\[
x \in A \Rightarrow x \in W_a \lor x \in W_c \Rightarrow x \in W_a,
\]

since \( x \in W_c \) yields \( x \in C \) which is disjoint from \( A \). Similarly \( x \in C \Rightarrow x \in W_c \).

Claim 2. For all \( x \),

\[
x \in A \leftrightarrow fx \in A, \quad x \in C \leftrightarrow fx \in C.
\]

This is trivial since \( f = [g(a, c)] \) and \((A, C) = (W_a, W_c) = (W_a \leq W_c, W_c < W_a)\).

Claim 3. For all \( x \),

\[
x \in A \leftrightarrow fx \in B, \quad x \in C \leftrightarrow fx \in D.
\]

The left-to-right implications follow from Claim 2. For the other direction, assume first that \( fx \in B \). A glance at the definition of \( W_a, W_c \) reveals that \( x \in W_a \) or \( x \in W_c \). The latter yields \( fx \in D \), contrary to assumption. Thus \( x \in W_a = A \). Similarly one shows \( fx \in D \) implies \( x \in C \). Q.E.D.

Obviously we can compose a reduction of \((X, Y)\) to \((A, C)\) with \( f \) to obtain a simultaneous reduction of any pair of disjoint r.e. sets to \((A, C)\) and \((B, D)\). A second corollary, noticed by J. R. Shoenfield, is this: For \( A, B, C, D \) as in the Theorem, any set \( X \) interpolated between \( A \) and \( B \), \( A \subseteq X \subseteq B \), has degree at least \( \theta \). [N.B. Without \( C \) and \( D \), this need not hold: Creative sets can have recursive interpolants.]

2. Some metamathematical applications. We give a few corollaries concerning the metamathematics of r.e. systems of arithmetic (for definiteness: extensions of Robinson's \( \mathbb{Q} \)) that were previously obtainable only via self-referential formulae (cf. Shepherdson 1960, Smoryński A).

Definitions. A formula \( \varphi x_0 \cdots x_{n-1} \) semirepresents a relation \( R \subseteq \omega^n \) in a theory \( \mathcal{T} \) iff, for all \( x_0, \ldots, x_{n-1} \),

\[
\mathcal{T} \vdash \varphi \overline{x}_0 \cdots \overline{x}_{n-1} \leftrightarrow Rx_0 \cdots x_{n-1}.
\]
\( \varphi \) dually semirepresents a disjoint pair of relations, \( R, S \) iff \( \varphi \) semirepresents \( R, S \), respectively. \( \varphi \) represents \( R \) iff \( \varphi \) dually semirepresents \( R \) and its complement. A formula \( \varphi v_0 \cdots v_n \) semirepresents (represents) a partial (total) function \( f \) iff (i) \( \varphi \) semirepresents (represents) the graph of \( f \), and (ii) \( \varphi \) satisfies a unicity condition, say,

\[ \vdash \varphi v_0 \cdots v_{n-1} v \land \varphi v_0 \cdots v_{n-1} v' \rightarrow v = v'. \]

[This is stronger than necessary for most purposes.]

**Corollary 1.** Let \( \mathcal{T} \) be a consistent r.e. extension of \( \mathcal{R} \). For any disjoint pair, \( R, S \) of \( n \)-ary r.e. relations, there is a formula \( \varphi v_0 \cdots v_{n-1} \in \Sigma_1 \) which dually semirepresents \( R, S \) in \( \mathcal{T} \); and, moreover, \( \varphi v_0 \cdots v_{n-1} \) defines \( R \) in the set of natural numbers.

**Proof.** Obviously we can assume the Theorem proven for \( n \)-ary relations. Moreover, by Smullyan's Dual Completeness Theorem, we can assume \( R, S \) to be effectively inseparable. So let \( \psi_0, \psi_1 \) be \( \Sigma_1 \) definitions of \( R, S \) and let \( A = R, C = S, B = \{ (x_0, \ldots, x_{n-1}) : \vdash (\psi_0 \iff \psi_1) \bar{x}_0 \cdots \bar{x}_{n-1} \} \), and \( D = \{ (x_0, \ldots, x_{n-1}) : \vdash (\psi_0 \iff \psi_1) \bar{x}_0 \cdots \bar{x}_{n-1} \} \). Now simply define

\[ \exists v_0 \cdots v_{n-1} [x v_0 \cdots v_{n-1} v_0' \cdots v_{n-1}' \land (\psi_0 \iff \psi_1) v_0' \cdots v_{n-1}'], \]

where \( \chi \in \Sigma_1 \) represents the recursive function \( f \) of the Theorem. Q.E.D.

The correctness of the semirepresentation of \( R \) is the novel feature of this proof. While it comes free with Shepherdson's proof via self-referential formulae, the correctness has either been lacking in recursion-theoretic proofs of Corollary 1 (Ehrenfeucht and Feferman 1960, Putnam and Smullyan 1960), or has resulted in non-\( \Sigma_1 \) semirepresentations (Hájková and Hájek 1972).

**Corollary 2.** The dual semirepresentation \( \varphi \) for disjoint \( R, S \) can be chosen uniformly in an r.e. sequence, \( \mathcal{T}_0, \mathcal{T}_1, \ldots \), of consistent extensions of \( \mathcal{R} \).

The proof is as before: Let \( B_i, D_i \) be the sets of tuples provably in, respectively out of, \( R \preceq S \) in \( \mathcal{T}_i \), and let \( B = \bigcup B_i \preceq \bigcup D_i \), \( D = \bigcup D_i \). Again, this result was originally quite easily proven by means of formal diagonalization.

**Corollary 3.** Let \( f \) be partial recursive; \( \mathcal{T}_0, \mathcal{T}_1, \ldots \) an r.e. sequence of consistent extensions of \( \mathcal{R} \). There is a formula \( \varphi v_0 \cdots v_n \in \Sigma_1 \) which correctly uniformly semirepresents \( f \) in each \( \mathcal{T}_i \). Moreover, we can assume

\[ \vdash \neg \varphi \bar{x}_0 \cdots \bar{x}_{n-1} \iff \exists z \neq y (f x_0 \cdots x_{n-1} = z). \]

Again the result is sharper than the original recursion-theoretic result (Ritchie and Young 1968/1969). We omit the proof.

As a final application we have

**Corollary 4.** Let \( \mathcal{T}_0 \preceq \mathcal{T}_1 \) be consistent r.e. extensions of \( \mathcal{R} \) and let
\( R_0 \subseteq R_1 \) be \( n \)-ary r.e. relations. There is a formula \( \varphi \) such that \( \varphi \) semirepresents \( R_1 \) in \( \mathfrak{T}_1 \).

**Proof.** We shall cheat slightly. Di Paola 1966 shows that there is a \( \psi_0 \) which semirepresents \( R_0 \) in \( \mathfrak{T}_0 \) and \( \omega^n \) in \( \mathfrak{T}_1 \). So let \( \psi_1 \) uniformly semirepresent \( R_1 \) in \( \mathfrak{T}_0 \), \( \mathfrak{T}_1 \) and define \( \varphi = \psi_0 \land \psi_1 \). Q.E.D.

Di Paola's full result required there to be a recursive interpolant between \( R_0 \) and \( R_1 \).

**References**

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