UNIQUE BALAYAGE IN FOURIER TRANSFORMS
ON COMPACT ABELIAN GROUPS

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Abstract. Let $K$ be a compact subset of the compact abelian group $G$ and let $\Lambda$ be a subset of the dual group $\Gamma$. Unique balayage is said to be possible for $(K, \Lambda)$ if, for every $\mu \in M(G)$, there is a unique $\nu \in M(K)$ whose Fourier transform, $\hat{\nu}$, agrees on $\Lambda$ with $\hat{\mu}$.

We prove that in order that there be any $K$ with unique balayage possible for $(K, \Lambda)$, $\Lambda$ must belong to the coset ring of $\Gamma$. The converse of this statement is false. Some examples are given for the case where $G$ is the circle group.

1. General results on unique balayage. Let $G$ be a compact abelian group (written multiplicatively) with dual group $\Gamma$ (written additively). If $K \subset G$ is compact and $\Lambda \subset \Gamma$ then, following Beurling [2], balayage is said to be possible for $(K, \Lambda)$, if, for every measure $\mu \in M(G)$ there is a measure $\nu \in M(K)$ with

$$\hat{\mu} (\lambda) = \hat{\nu} (\lambda) \quad \text{for all } \lambda \in \Lambda.$$ 

Here $\hat{\mu}$ denotes the (inverse) Fourier transform

$$\hat{\mu} (\gamma) = \int_G \gamma(x) \, d\mu(x).$$

We would like to thank Professor Colin Graham for acquainting us with the following problem of Professor S. Hartman. Determine whether there are nontrivial examples of sets $K$ and $\Lambda$ with $K \subset T$, the circle group, such that

(1) balayage is possible for $(K, \Lambda)$ and
(2) if $\mu \in M(K)$ and $|\mu|_{\Lambda} = 0$ then $\mu = 0$.

In any $G$ and $\Gamma$, if these conditions are satisfied we shall say that unique balayage is possible for $(K, \Lambda)$.

We recall that the coset ring of $\Gamma$ is the smallest algebra of sets containing all cosets of subgroups of $\Gamma$. Our basic result is

Theorem 1. Given $\Lambda \subset \Gamma$, in order that there be some $K \subset G$ with unique balayage possible for $(K, \Lambda)$ it is necessary that $\Lambda$ belong to the coset ring of $\Gamma$.

Remark. We shall see in §2 that, when $\Gamma = Z$, there are some infinite $\Lambda$ in
the coset ring for which unique balayage is possible and others for which it is not.

**Proof of Theorem 1.** We first introduce some Banach spaces and reformulate the definition of unique balayage.

$B(\Lambda)$ is the space of restrictions to $\Lambda$ of Fourier transforms and is normed by

$$\|\phi\|_{B(\Lambda)} = \inf\{\|\mu\| : \mu \in M(G) \text{ and } \hat{\mu}|\Lambda = \phi\}.$$  

$C_{\Lambda}(G)$ is the space of continuous functions $f$ on $G$ such that $\hat{f}(-\gamma) = 0$ for all $\gamma \in \Gamma \setminus \Lambda$. Here, by $\hat{f}$ we again mean the inverse transform

$$\hat{f}(\gamma) = \int_G f(x)\gamma(x) \, dx$$

the integral being taken with respect to normalized Haar measure. $C_{\Lambda}(G)$ is given the uniform norm. It follows from the existence of a version of Cesàro summability on $G$ (see, for example, [1, p. 56]) that $C_{\Lambda}(G)$ is the uniform closure of the set of trigonometric polynomials with frequencies in $\Lambda$. The pairing of $\phi \in B(\Lambda)$ and $f \in C_{\Lambda}(G)$ given by

$$\langle \phi, f \rangle = \int_G f(x) \, d\mu,$$

where $\mu \in M(G)$ is any measure with $\hat{\mu}|\Lambda = \phi$, represents $B(\Lambda)$ as the dual space of $C_{\Lambda}(G)$. (Use the Hahn-Banach and Riesz theorems; see, for example, [4, p. 116] and note that, since $G$ is compact, $A\mathcal{P}_{\Lambda}(G) = C_{\Lambda}(G)$.)

$C(K)$ denotes the space of complex-valued continuous functions on $K$.

We define a bounded linear operator $S: C_{\Lambda}(G) \to C(K)$ by

$$Sf = f|K. \quad (1)$$

Regarding $M(K)$ as the dual space of $C(K)$, we see that the adjoint $S^*: M(K) \to B(\Lambda)$ is given by

$$S^*\mu = \hat{\mu}|\Lambda. \quad (2)$$

Indeed, setting $\phi = \hat{\mu}|\Lambda \in B(\Lambda)$, we have, for all $f \in C_{\Lambda}(G)$,

$$\langle \phi, f \rangle = \int_K f(x) \, d\mu(x) = \langle \mu, Sf \rangle$$

so that we must have $\phi = S^*\mu$.

From (2) we see that unique balayage is possible for $(K, \Lambda)$ if and only if $S^*$ is bijective. Supposing this to be the case, it follows [3, p. 479] that $S$ is bijective and, therefore, invertible. Indeed, for $f \in C(K)$, $S^{-1}f$ is the unique $g \in C_{\Lambda}(G)$ with $g|K = f$.

Define $P: C(G) \to C_{\Lambda}(G)$ by

$$Pf = S^{-1}(f|K).$$

From the above, it follows that $Pf$ is the unique $g \in C_{\Lambda}(G)$ such that $g|K = f|K$. In particular, $P$ is a bounded projection of $C(G)$ onto $C_{\Lambda}(G)$. Theorem 1, therefore, follows from
Theorem 2. A necessary and sufficient condition for the existence of a bounded projection $P$ from $C(G)$ onto $C_A(G)$ is that $A$ belong to the coset ring of $\Gamma$.

Proof. This result and the argument establishing it are very similar to a result of Rosenthal [5, Theorem 3] about projections onto translation-invariant subspaces of $L^1(G)$. Some of the details differ, however.

Suppose, first, that such a $P$ exists. Write $f_t$ for the translate, $f_t(x) = f(xt)$, and define an operator $Q$ on $C(G)$ by

$$(Qf)(x) = \int_G (Pf_t)(xt^{-1}) \, dt$$

(integral with respect to Haar measure).

Since $G$ is compact, $t \to f_t$ is norm continuous so that the integrand above is continuous in $t$. If $\gamma \in \Gamma$ then $\gamma \in C(G)$ and $\gamma_t = \gamma(t)\gamma$. Thus

$$(Q\gamma)(x) = \int_G \gamma(t)(P\gamma)(xt^{-1}) \, dt = \int_G \gamma(xt^{-1})(P\gamma)(t) \, dt = \gamma(x)(P\gamma)^\ast (-\gamma).$$

Now, $P\gamma \in C_A(G)$ so that, if $\gamma \notin A$ then $Q\gamma = 0$. On the other hand, if $\gamma \in A$ then $P\gamma = \gamma$ so that $Q\gamma = \gamma$ also. Thus, $Q$ maps trigonometric polynomials into trigonometric polynomials belonging to $C_A(G)$. Since we clearly have $\|Qf\|_\infty \leq \|P\| \|f\|_\infty$, the fact that every function in $C(G)$ can be uniformly approximated by trigonometric polynomials now implies that, for $f \in C(G)$, $Qf$ is continuous and, hence, that $Qf \in C_A(G)$. (Thus, $Q$ is the “obvious” projection of $C(G)$ onto $C_A(G)$.)

Now, if we regard $Q$ as an operator from $C(G)$ to itself, then $Q^\ast : M(G) \to M(G)$. If $\mu \in M(G)$, then,

$$(Q^\ast \mu)^\ast (\gamma) = \langle \gamma, Q^\ast \mu \rangle = \langle Q\gamma, \mu \rangle$$

which is $\hat{\mu}(\gamma)$ or 0 according as $\gamma \in A$ or $\gamma \notin A$.

In particular, if $\delta$ is the unit point mass at 1, and if $\alpha = Q^\ast \delta$ then $\hat{\alpha}$ is the characteristic function of $A$ so that $\alpha$ is idempotent. By Cohen’s general result on idempotent measures, [6, Theorem 3.1.3], $A$ must belong to the coset ring of $\Gamma$.

Conversely, if $A$ belongs to the coset ring of $\Gamma$, then so does $-A$ and there is a measure $\alpha \in M(G)$ such that $\hat{\alpha}$ is the characteristic function of $-A$. In this case, if we set $Pf = f \ast \alpha$, then $P$ is a bounded projection of $C(G)$ onto $C_A(G)$.

2. Examples in the circle group. In general, if $A$ is finite, then unique balayage is possible for $(K, A)$ if and only if $K$ has the same cardinality as $A$ and the characters in $A$, considered as functions on $K$, are linearly independent (for then, $C_A(G) \mid K$ has the same dimension as $C(K)$).

For less trivial results, we turn to the case $G = T$, the circle group (realized as $\{z : |z| = 1\}$) so that $\Gamma = Z$. We note that $A$ belongs to the coset ring of $Z$
if and only if $\Lambda$ differs from a periodic set by at most a finite number of elements. We present, first, a class of examples showing that the converse to Theorem 1 fails.

Let $n > 1$ be an integer. Let $B$ be a nontrivial, proper subset of \{0, 1, 2, \ldots, n - 1\}, containing $b$ elements. Let $n\mathbb{Z}$ be the subgroup of multiples of $n$. Set

$$\Lambda = \bigcup_{j \in B} (j + n\mathbb{Z}).$$

**Theorem 3.** With $\Lambda$ as above, there is no compact $K \subset T$ with unique balayage possible for $(K, \Lambda)$.

**Proof.** Let $\omega = \exp(2\pi i/n)$ and let $\delta(z)$ denote the unit point mass at $z \in T$.

Set, for $j = 0, 1, 2, \ldots, n - 1$,

$$\mu_j = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-k} \delta(\omega^k).$$

Then

$$\hat{\mu}_j(m) = \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{j+m})^k$$

which is 1 or 0 according as $j + m \in n\mathbb{Z}$ or not. Thus, $\hat{\mu}_j$ is the characteristic function of the coset $-j + n\mathbb{Z}$. Let $D$ be the complement of $B$ in \{0, 1, 2, \ldots, n - 1\} so that $D$ has $d = n - b$ elements. In order for $f$ to belong to $C_\Lambda(T)$ it is, therefore, necessary and sufficient that

$$f \ast \mu_j = 0 \text{ for all } j \in D.$$

More explicitly, $f \in C_\Lambda(T)$ if and only if, for each $z \in T$,

$$\sum_{k=0}^{n-1} \omega^k f(z\omega^{-k}) = 0 \text{ for all } j \in D. \quad (3)$$

Let $H$ be the subgroup \{1, $\omega$, $\omega^2$, \ldots, $\omega^{n-1}$\} of $T$. For each $z \in T$, (3) is a set of relations which $f|zH$ must satisfy. Since the $n \times n$ matrix whose $jk$th element is $\omega^k$ has orthogonal rows the $d$ relations in (3) are linearly independent. Thus, $C_\Lambda(T)|zH$ has dimension $n - d = b$. Since $C(K)|zH$ has as its dimension the cardinality of $K \cap zH$, in order that the operator $S$ (in equation (1)) be bijective it is, therefore, necessary that for each $z \in T$, $K \cap zH$ have exactly $b$ elements. Equivalently, each $z \in T$ must belong to exactly $b$ of the translates $\omega^kK$, $k = 0, 1, 2, \ldots, n - 1$.

No compact set $K$ can have this property. Indeed, for each $W \subset H$ which has $b$ elements, define

$$K_W = \bigcap_{z \in W} zK.$$

If $W_1 \neq W_2$ then any element in $K_{W_1} \cap K_{W_2}$ would belong to at least $b + 1$ of the $\omega^kK$. Thus, the various $K_W$ are disjoint. Clearly, no $K_W$ is all of $T$ but
we must have
\[ T = \bigcup_w K_w. \]
Since each \( K_w \) is compact, this contradicts the connectedness of \( T \) and so proves Theorem 3.

For some infinite sets in the coset ring of \( Z \), unique balayage is possible, as the following example shows. Let \( \Lambda = 2\mathbb{Z} \cup \{1\} \) and let \( K = \{e^{\theta i}: 0 < \theta < \pi\} \). Note that a continuous \( g \) belongs to \( C_{2\mathbb{Z}}(T) \) if and only if \( g(e^{\theta i}) \) has period \( \pi \).

Suppose \( f \in C(K) \). Set
\[ g(z) = f(z) + \frac{1}{2}(f(-1) - f(1))z \]
for \( z \in K \). Then \( g(1) = g(-1) \) so that \( g \) may be extended to a function \( h \in C_{2\mathbb{Z}}(T) \). Then, if
\[ f_1(z) = h(z) - \frac{1}{2}(f(-1) - f(1))z \]
we see that \( f_1 \in C_A(T) \) and that \( f_1|K = f \) so that \( S \) as defined in equation (1) is onto.

On the other hand, suppose \( f \in C_A(T) \) and \( f|K = 0 \). If
\[ h(z) = f(z) - \hat{f}(1)z \]
then \( h \in C_{2\mathbb{Z}}(T) \) so that \( h(e^{\theta i}) \) has period \( \pi \). Thus
\[ \hat{h}(2m) = \int_0^\pi h(e^{\theta i})e^{2im\theta} \frac{d\theta}{\pi} = \int_0^\pi f(e^{\theta i})e^{2im\theta} \frac{d\theta}{\pi} - \int_0^\pi \hat{f}(1)e^{i(2m+1)\theta} \frac{d\theta}{\pi} = \frac{-2i}{\pi(2m+1)} \hat{f}(-1) \]
because \( f|K = 0 \). If \( \hat{f}(-1) \neq 0 \), these are the Fourier coefficients of a discontinuous step function. Thus \( \hat{f}(-1) \) and, hence, \( \hat{h} \) and \( \hat{f} \), must vanish. Thus, \( f = 0 \) so that unique balayage is possible for \((K, \Lambda)\).

**BIBLIOGRAPHY**