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MULTIPLIERS ON COMPACT GROUPS

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ABSTRACT. We give some sufficient conditions for a function on a compact totally disconnected abelian group to be an L^p Fourier multiplier.

1. Introduction. Let X denote a compact abelian group with a strictly decreasing sequence of open compact subgroups $\{X_n\}_0^\infty$ such that $\bigcup X_n = X$ $\bigcap X_n = \{0\}$ and $2 \leqslant |X_n| \cdot |X_{n+1}|^{-1} \leqslant b$, where $|S|$ denotes the Haar measure of a set S. Let G denote the dual of X and G_n the annihilator of X_n in G; thus $\{G_n\}$ is an increasing sequence of open compact subgroups of G, $\bigcup G_n = G$, $\bigcap G_n = \{0\}$ and $2 \leqslant |G_{n+1}| \cdot |G_n|^{-1} \leqslant b$. We denote by $d\chi$ and dx the Haar measures on X and G respectively, and assume that these are adjusted so that the inversion theorem holds.

If $\phi \in L^\infty(X)$ then ϕ defines a bounded linear operator T_ϕ on $L^2(G)$ via the formula

$$(T_\phi f)^\wedge = \phi \hat{f}.$$

We say that ϕ is an L^2 Fourier multiplier. Similarly for $1 \leqslant p \leqslant \infty$, we say that ϕ is an L^p Fourier multiplier, (and write $\phi \in M_p(X)$), if there exists a number B such that

$$\|T_\phi f\|_p \leqslant B \|f\|_p \tag{1}$$

for all f in $L^p \cap L^2(G)$; we write $\|\phi\|_{M_p}$ for the smallest value of B for which (1) holds. It is well known that $M_p = M_{p'}$ (where, as always $(1/p) + (1/p') = 1$) and

$$A(X) = M_1(X) \subseteq M_p(X) \subseteq M_q(X) \subseteq M_2(X) = L^\infty(X)$$

when $1 \leqslant p \leqslant q \leqslant 2$ and where $A(X)$ is the space of Fourier transforms of integrable functions with the inherited norm. A wealth of information about multipliers is contained in the book [1] of Edwards and Gaudry.

2. Suppose $1 < \theta < \infty$; for $n \geqslant 0$ we define the subgroup X_n^θ of X by the formula

$$X_n^\theta = X_j \quad \text{where } |X_j| \geqslant |X_n|^\theta > |X_{j+1}|. \tag{2}$$

Our main result is the following:

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THEOREM. *Suppose ϕ is a function on X constant on cosets of X_n^θ outside X_n. If*

$$|\phi(\chi)| \leqslant B|X_n|^{(\theta-1)/2} \quad \text{when} \quad \chi \in X_{n-1} \setminus X_n \tag{3}$$

for some constant B independent of n, then $\phi \in M_p(X)$ for $1 < p < \infty$.

The proof employed is singular-integral in spirit, although no use is made of Calderón-Zygmund type covering lemmas. We need the following:

LEMMA. *For $1 < \theta < \infty$ we define the subgroup G_n^θ of G by the formula*

$$G_n^\theta = G_j \quad \text{where } |G_j| \leqslant |G_n|^\theta < |G_{j+1}|. \tag{4}$$

In other words, G_n^θ is the annihilator of X_n^θ. Suppose k is an integrable function on G, constant on cosets of G_n outside G_n^θ. If

$$|\hat{k}(\chi)| \leqslant B|X_n|^{(\theta-1)/2+\beta} \quad \text{when } \chi \in X_{n-1} \setminus X_n \tag{5}$$

for some $\beta > 0$, then

$$\|k * f\|_\infty \leqslant B \cdot C \cdot \|f\|_\infty$$

where C is a constant independent of $\|k\|_1$.

PROOF. Fix f in L^∞. By translation invariance it suffices to show that, for every $N \geqslant 0$,

$$\left| |G_N|^{-1} \int_{G_N} k * f \, dx \right| \leqslant B \cdot C\|f\|_\infty.$$

Fix N and write $f = f_1 + f_2$ where $f_1 = f \cdot \xi_{G_{N+1}^\theta}$ (ξ_S denotes the indicator function of the set S). Then

$$\begin{aligned}
\left| |G_N|^{-1} \int_{G_N} k * f_1 \, dx \right| &= |D_N * k * f_1(0)| \quad \text{where } D_n = \xi_{G_n} \cdot |G_n|^{-1} \\
&= \left| \int_{X_N} \hat{k}\hat{f}_1 \, d\chi \right| \\
&\leqslant B|X_N|^{(\theta-1)/2+\beta} \int_{X_N} |\hat{f}_1| \, d\chi \quad \text{by (5)} \\
&\leqslant B|X_N|^{(\theta-1)/2+\beta+1/2} \left(\int_{X_N} |\hat{f}_1|^2 \, d\chi \right)^{1/2}
\end{aligned}$$

by Hölder's inequality,

$$\begin{aligned}
&\leqslant B|X_N|^{\theta/2+\beta}\|f_1\|_2 \\
&\leqslant B|X_N|^{\theta/2+\beta}|G_{N+1}|^{\theta/2}\|f\|_\infty
\end{aligned}$$

by the definition of f_1,

$$\leqslant B \cdot b^{\theta/2}|X_N|^\beta\|f\|_\infty. \tag{6}$$

Now clearly,

$$\left| |G_N|^{-1} \int_{G_N} k * f_2 \, dx \right|$$
$$\leqslant |G_N|^{-1} \int_{G_N} \left| k * f_2 - |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_2 \right| dx$$
$$+ \left| |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_2 \right|.$$

But

$$|G_N|^{-1} \int_{G_N} \left| k * f_2 - |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_2 \right| dx$$
$$\leqslant |G_N|^{-1} \int_{G_N} |k * f_2 - \sigma| \, dx + |G_{N+1}|^{-1} \int_{G_{N+1}} |k * f_2 - \sigma| \, dx$$
$$\text{where } \sigma = \int_G k(-y) f_2(y) \, dy. \tag{7}$$

The second term on the right of (7) is equal to

$$|G_{N+1}|^{-1} \int_{G_{N+1}} dx \left(\left| \int_{G \setminus G^\theta_{N+1}} (k(x-y) - k(-y)) f_2(y) \, dy \right| \right) = 0$$

since $x \in G_{N+1}$ and k is constant on the cosets of G_{N+1} outside G^θ_{N+1}. The same argument shows that the first term on the right of (7) is also zero, so

$$\left| |G_N|^{-1} \int_{G_N} k * f_2 \, dx \right| \leqslant \left| |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_2 \, dx \right|.$$

Now write $f_2 = f_3 + f_4$ where $f_3 = f_2 \cdot \xi_{G^\theta_{N+2}}$. The argument used to estimate $|G_N|^{-1} \int_{G_N} k * f_1 \, dx$ shows that

$$\left| |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_3 \, dx \right| < B \cdot b^{\theta/2} |X_{N+1}|^\beta \|f\|_\infty,$$

and the argument used to estimate $|G_N|^{-1} \int_{G_N} k * f_2 \, dx$ shows that

$$\left| |G_{N+1}|^{-1} \int_{G_{N+1}} k * f_4 \, dx \right| \leqslant \left| |G_{N+2}|^{-1} \int_{G_{N+2}} k * f_4 \, dx \right|.$$

We may suppose without loss of generality that $\hat{k} = 0$ on X_M for some large M; thus a continuation of the above argument leads to the estimate

$$\left| |G_N|^{-1} \int_{G_N} k * f \, dx \right| \leqslant b^{\theta/2} \cdot B \cdot \|f\|_\infty \left(\sum_N^M |X_n|^\beta \right)$$
$$\leqslant b^{\theta/2} \cdot B \cdot \|f\|_\infty \sum_0^\infty 2^{-n\beta}$$
$$\leqslant b^{\theta/2} \cdot B \cdot (1 - 2^{-\beta})^{-1} \|f\|_\infty,$$

which proves the lemma.

PROOF OF THEOREM. Set $\Theta(\chi) = |X_n|^{(\theta-1)/2}$ when $\chi \in X_{n-1} \setminus X_n$, and consider the family of operators U_z on $L^2(G)$ defined by

$$(U_z f)\hat{} = \hat{f} \cdot \phi \cdot \Theta^{-z+\beta}$$

where $0 \leqslant \mathrm{Re}' z \leqslant 1$ and $\beta > 0$. It is easy to check that, by virtue of (3), the mapping $z \to U_z$ is uniformly bounded and strongly continuous in the strip $0 \leqslant \mathrm{Re}' z \leqslant 1$ and analytic in $0 < \mathrm{Re}' z < 1$, to the space of bounded linear operators on $L^2(G)$.

It follows immediately from (3) that

$$\|U_{1+iy} f\|_2 \leqslant B \cdot \|f\|_2. \tag{8}$$

Furthermore it follows from the above lemma that

$$\|U_{iy} f\|_\infty \leqslant B \cdot C \cdot \|f\|_\infty. \tag{9}$$

(This amounts essentially to the observation that a function constant on cosets of X_n^θ outside X_n has Fourier transform constant on cosets of G_n outside G_n^θ.) An application of Stein's interpolation theorem, [**6**, p. 205], shows that

$$\phi \cdot \Theta^{-t+\beta} \in M_{2/t}(X)$$

where $0 < t < 1$, and $\beta > 0$. To see that $\phi \in M_p$ for all p in $(1, \infty)$ fix p in $(2, \infty)$ and set $t = 2/p$. Then

$$\phi\Theta^{-2/p+\beta} \in M_p(X),$$

in particular when $\beta = 2/p$. The proof is complete.

Virtually the same proof yields:

COROLLARY. *Under the same conditions as stated in the theorem*

$$\phi \cdot \Theta^{-t} \in M_p(X) \quad \textit{when } 2/(2-t) < p < 2/t.$$

It is not difficult, using the ideas in [**2**], to construct examples of functions ϕ such that $\phi \cdot \Theta^{-t} \notin M_p$ when $p > 2/t$. The interesting case is, of course, when $p = 2/t$–see remark (b) below.

3. **Remarks.** (a) The condition that $|X_n| \cdot |X_{n+1}|^{-1} \leqslant b$ is not really necessary. If $b_n = |X_n| \cdot |X_{n+1}|^{-1}$ is such that $b_n \uparrow \infty$ and $\sum b_n^\beta < \infty$, then the above proof is easily adapted to show that:

If ϕ is a function constant on cosets of X_n outside X_n then $|\phi(\chi)| \leqslant B \cdot b_n^{-(1/2+\beta)}$ when $\chi \in X_n \setminus X_{n+1}$ for some $\beta > 0$ implies that $\phi \in A(X)$, and $|\phi(\chi)| \leqslant B \cdot b_n^{-1/2}$ when $\chi \in X_n \setminus X_{n+1}$ implies that $\phi \in M_p(X)$ for $1 < p < \infty$.

This complements some results of Spector [**5**]. Further results may be obtained by considering subsequences $\{X_{n_k}\}$ of $\{X_n\}$.

(b) There is a natural definition of BMO(G), the space of functions of bounded mean oscillation on G, see [**2**]. It would be interesting to know if the functions ϕ satisfying the hypotheses of the theorem are $L^\infty \to$ BMO "multipliers". A positive answer would imply that $\phi \cdot \Theta^{-t} \in M_{2/t}$ when $0 < t < 1$.

(c) Let Z denote the group of integers, and Z_n the subgroup $2^n \cdot 2^{n-1} \cdot \cdots \cdot 2^1 Z$, where $n \geqslant 0$. Suppose ϕ is a function constant on cosets of Z_{n+1} in $Z_n \setminus Z_{n+1}$. If $|\phi(m)| \leqslant B \cdot 2^{-(n+1)(\beta+1/2)}$ when $m \in Z_n \setminus Z_{n+1}$ for some $\beta > 0$ then ϕ is a Fourier-Stieltjes transform, and if $|\phi(m)| < B \cdot 2^{-(n+1)/2}$ when $m \in Z_n \setminus Z_{n+1}$ then $\phi \in M_p(Z)$ for $1 < p < \infty$. To see this argue as follows: Let $X = \Delta_{\mathbf{a}}$, the **a**-adic integers, where $\mathbf{a} = (2, 4, 8, 16, \ldots)$ (see [**3**, §10]), and let $X_n = \{x = (x_j)_0^\infty \in X: x_j = 0 \text{ when } 0 \leqslant j \leqslant n-1\}$. It is easily seen that $|X_n| = (2^n \cdot 2^{n-1} \cdot \cdots \cdot 2^1)^{-1} \cdot X$ has an (algebraic!) subgroup isomorphic to Z, namely the group generated by the element $(1, 0, 0, 0, \ldots)$, which we also denote by Z. Some rather tedious calculations show that the cosets of Z_{n+1} in $Z_n \setminus Z_{n+1}$ sit inside the cosets of X_n in $X_n \setminus X_{n+1}$; hence ϕ is the restriction of a function Φ on X, satisfying the hypotheses in (a) above. The result now follows immediately from well-known results about restrictions of Fourier multipliers to subgroups, see for example Saeki [**4**, Corollary 4.6].

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