MARKOV PROPERTY OF EXTREMAL LOCAL FIELDS

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Abstract. We show that extremal local field on \((E, \mathcal{S})^T\), with \(T = \mathbb{Z}\) or \(\mathbb{R}\) and \((E, \mathcal{S})\) standard, possesses the Markov property. This result generalizes that of F. Spitzer in the case \(T = \mathbb{Z}\), \(E\) countable and a result of G. Royer and M. Yor on extremal measures associated to certain diffusion processes.

I. Introduction. Let \(T = \mathbb{R}\) or \(\mathbb{Z}\), let \((E, \mathcal{S})\) be a standard Borel space, \((\Omega, \mathcal{F}, P)\) a probability space and \((X_i)_{i \in T}\) a stochastic process taking values in \(E\). If \(\Lambda\) is a subset of \(T\), we note \(\mathcal{A}_\Lambda\) the sub-\(\sigma\)-algebra of \(\mathcal{F}\) generated by \(X_i, i \in \Lambda\).

The process \((X_t)\) is called a local Markov process (or local Markov field) if for every \(a < b\) in \(T\), every positive \(\mathcal{F}_{(a,b)}\)-measurable function \(f\) we have
\[
E\left[ f | \mathcal{A}_{(a,\infty)} \right] = E\left[ f | \mathcal{A}_{(a,b)} \right] \quad \text{P-a.s.}
\]

The process \((X_t)\) is called a Markov process (or Markov field) if for every \(t\) in \(T\), every positive \(\mathcal{F}_t\)-measurable function \(f\) we have
\[
E\left[ f | \mathcal{A}_{(\infty,t]} \right] = E\left[ f | \mathcal{F}_t \right] \quad \text{P-a.s.}
\]

The property (0) is called local Markov property, the property (1) is called Markov property. It is known that Markov property implies local Markov property (see for example [1]) but the converse is false (cf. [4], [5]).

The purpose of this paper is to show that under quite general conditions the local Markov property and the hypothesis \(\mathcal{A}_\infty = \{\emptyset, \Omega\}\) P-a.s. imply the Markov property, where \(\mathcal{A}_\infty = \bigcap_{t \in T} \mathcal{A}_{[t,\infty]}\) is the asymptotic \(\sigma\)-algebra of the process (we note similarly \(\mathcal{A}_{-\infty} = \bigcap_{t \in T} \mathcal{A}_{(-\infty,t]}\)).

This result has been proved under the hypotheses \(E\) countable and \(T = \mathbb{Z}\) by F. Spitzer (cf. [5]). In the case \(E = \mathbb{R}\), \(T = \mathbb{R}\), G. Royer and M. Yor have proved this result for certain diffusion processes associated to one-dimensional quantum fields (cf. [4]).

II. Sufficient conditions for Markov property.

Notations. We take the path space \((\Omega, \mathcal{A}) = (E, \mathcal{S})^T\), \(X_t(\omega) = \omega_t, \forall \omega \in E^T, \forall t \in T\). If \(\mu\) is a probability on \((\Omega, \mathcal{A})\), and \(\Lambda \subset T\), we note \(\mu_\Lambda\) the

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restriction of μ on $\mathcal{G}_\Lambda$ and $E_\mu[g|\mathcal{G}_\Lambda]$ the conditional expectation of a random variable $g$ on $\Omega$ with respect to the sub-$\sigma$-algebra $\mathcal{G}_\Lambda$.

If a probability $\nu$ on a $\sigma$-algebra $\mathcal{B}$ is absolutely continuous with respect to another probability $\nu'$ on $\mathcal{B}$, we write $\nu \ll \nu'$.

We say that $\mu$ is a local Markov field (resp. Markov field) if $(X_t)$ is a local Markov process (resp. Markov process) under $\mu$.

The main result of the paper is

**Theorem 1.** Let $T = \mathbb{R}$ or $\mathbb{Z}$, $(E, \mathcal{S})$ a standard Borel space. Let $\mu$ be a local Markov field on $(\Omega, \mathcal{G})$ such that

(i) For every $t \in T$, there exist $t' < t < t''$ in $T$ and $\sigma$-finite measures $\nu_t$, $\nu_{t'}$, $\nu_{t''}$ on $(E, \mathcal{S})$ such that

$$\mu_{t',t,t''} \ll \nu_t \otimes \nu_{t'} \otimes \nu_{t''}.$$ 

(ii) $\mathcal{G}_\infty$ (resp. $\mathcal{G}_{-\infty}$) is trivial for $\mu$.

Then $\mu$ is a Markov process.

We give the proof for the case $T = \mathbb{R}$ (the case $T = \mathbb{Z}$ is similar) under the hypothesis $\mathcal{G}_\infty$ trivial (for the case $\mathcal{G}_{-\infty}$ trivial, we use the additional remark that the Markov property is symmetric with respect to the direction of time).

**Proof.** Let $t \in T$, $t' < t < t''$, $\nu_t$, $\nu_{t'}$, $\nu_{t''}$ as in (i), let $k$ be a positive integer greater than $t''$, $S = \{t', t, t'', k, k + 1, k + 2, \ldots\}$.

(a) On $(E, \mathcal{S})^S = (E, \mathcal{S}) \times (E, \mathcal{S})^{S_1}$ we have $\mu_S \ll \nu_t \otimes \mu_{S_1}$.

Let $(E, \mathcal{S})^S = (E, \mathcal{S})^{(t)} \times (E, \mathcal{S})^{(t',t'')}$ and $(E, \mathcal{S})^{(k,k+1,\ldots)}$,

$$(E, \mathcal{S})^S = (X, \mathcal{S}) \times (Y, \mathcal{S}) \times (Z, \mathcal{S}).$$

Consider the measure $\mu_{(t',t,t'')}$ on $X \times Y = E^{(t)} \times E^{(t',t'')}$ and let $\mu^{(x,x',x'')}(F)$, for $(x, x', x'') \in Y \times Z$, $F \in \mathcal{S}$ be the transition probability on $Y \times \mathcal{S}$ such that

$$\mu_{(t',t,t'')}(x,y) = \int_Y \mu^{(x,x',x'')}(y) \, d\mu_{(t',t,t'')}(x,y).$$

(cf. [3, Proposition V.4.4, p. 183]).

And similarly on $X \times (Y \times Z)$ let $\mu^{(x,x',x',x'',x'',x'',\ldots)}(F)$, for $(x, x', x'', x_k, x_{k+1}, \ldots) \in Y \times Z$, $F \in \mathcal{S}$ be the transition probability on $(Y \times Z) \times \mathcal{S}$ such that

$$\mu_S = \int_{Y \times Z} \mu^{(x,x',x',x'',x'',x'',\ldots)}(x,y) \, d\mu_{S_1}(x,x',x'',x_k, x_{k+1}, \ldots).$$

By local Markov property (0), we have

$$\mu^{(x,x',x'',\ldots)} = \mu^{(x,x')}, \quad \text{for } \mu_{S_1} \text{-almost all } (x', x'').$$

But by (i), we have $\mu_{(t',t,t'')} \ll \nu_t \otimes \nu_{t'} \otimes \nu_{t''}$ and $\mu_{(t',t'')} \ll \nu_t \otimes \nu_{t''}$, so relation (2) implies

$$\mu^{(x,x',x'',\ldots)} \ll \nu_t, \quad \text{for } \mu_{S_1} \text{-almost all } (x', x'').$$

Comparing (4) and (5), we deduce that

$$\mu^{(x,x',x',x'',x'',\ldots)} \ll \nu_t, \quad \text{for } \mu_{S_1} \text{-almost all } (x', x'', x_k, \ldots).$$
Combining this relation and (3), we obtain

$$
\mu_S \ll \nu_t \otimes \mu_{S\setminus t}.
$$

(b) We have $\mathcal{A}_t = \lim_{n \to \infty} \mathcal{A}_n \lor (\bigvee_{m \geq n} \mathcal{A}_m)$: Let $(X, \mathcal{X}) = (E, \mathcal{E})^{(t)}, (W, \mathcal{W}) = (E, \mathcal{E})^{(t)}$, then $(E, \mathcal{E})^S = (X, \mathcal{X}) \times (W, \mathcal{W})$.

Let $\mathcal{B}_n = \bigvee_{m \geq n} \mathcal{A}_m$, for $n \geq k$, we shall show that $\lim_{n \to \infty} \mathcal{A}_t \lor \mathcal{B}_n = \mathcal{A}_t$, $\mu_S$-a.e. But we remark that $\mathcal{A}_t = \mathcal{X}$, $-\mathcal{B}_n \subset \mathcal{W}$, $\forall n \geq k$ and $\lim_{n \to \infty} \mathcal{B}_n = \mathcal{X}$ and $\mathcal{W}$ are independent with respect to $\nu_t \otimes \mu_{S\setminus t}$.

Therefore

$$
\lim_{n \to \infty} \mathcal{A}_t \lor \mathcal{B}_n = \mathcal{A}_t, \quad \nu_t \otimes \mu_{S\setminus t}-a.e.
$$

and also $\mu_S$-a.e. since $\mu_S \ll \nu_t \otimes \mu_{S\setminus t}$.

(c) Now let $f > 0$, $\mathcal{A}_{(a,b)}$-measurable with $t < a < b$, then (1) is satisfied:

Let $n > \beta$ with $n$ positive integer, by local Markov property (0), we have

$$
E_{\mu}[f | \mathcal{A}_{[t-\infty, t]}|] = E_{\mu}[f | \mathcal{A}_{(t,n)}] = E_{\mu}[f | \mathcal{A}_t].
$$

So by (b) and martingale theorem, we have

$$
\lim_{n \to \infty} E_{\mu}[f | \mathcal{A}_{[t-\infty, t]}|] \to E_{\mu}[f | \mathcal{A}_t].
$$

Taking the conditional expectation with respect to $\mathcal{A}_{(1-\infty, a]}$ of the both sides, we obtain

$$
E_{\mu}[f | \mathcal{A}_{[t-\infty, a]}] = E_{\mu}[f | \mathcal{A}_t], \quad \mu-a.e.
$$

(d) From (c) we deduce (1) since the $\mathcal{A}_{[a,b)}$, $t < a < b$, generate $\mathcal{A}_{(t, \infty)}$.

Remark. It is well known that the conditions $\mathcal{B}_n \setminus \{\emptyset, \Omega\}$ do not imply $\mathcal{A}_t \lor \mathcal{B}_n \lor \mathcal{A}_t$ without extra hypotheses.

III. Application to statistical mechanics and one dimensional field theory.

Local Markov specifications. Let $T = \mathbb{R}$ or $\mathbb{Z}$, $(E, \mathcal{E})$ be a standard Borel space, $(\Omega, \mathcal{F}) = (E, \mathcal{E})^T$.

We call local Markov specifications (see [2]) the given of a family $\pi = (\pi_{[a,b]}(\omega, A) (\omega \in \Omega, A \subset \mathcal{E})$ such that

(i) For every $t \in T$ there exist $a < t < b$, a $\sigma$-finite measure $\nu_t$ on $(E, \mathcal{E})$

such that the restriction of $\pi_{[a,b]}(\omega, A)$ to $\mathcal{A}_t$ is absolutely continuous with respect to $\nu_t, \forall \omega \in \Omega$.

(ii) $\pi_{[a,b]}(\cdot, A)$ is $\mathcal{E}_{[t-\infty,a]} \lor [b, + \infty]$-measurable, $\forall A \subset \mathcal{E}$.

(iii) If $[a, b] \subset [c, d]$, we have $\pi_{[c,d]}(\mathcal{A}_{[a,b]}(\omega, \cdot)) = \pi_{[c,d]}(\mathcal{A}_{[a,b]}(\omega, \cdot))$ where

$$
\pi_{[c,d]}(\omega, A) = \int_\pi \pi_{[c,d]}(\omega, d\omega') \pi_{[a,b]}(\omega', A), \quad \forall A \subset \mathcal{E}, \forall \omega \in \Omega.
$$

We call local Markov field (or Gibbs state) specified by $\pi$ every probability $\mu$ on $(\Omega, \mathcal{F})$ satisfying

$$
E_{\mu}[A | \mathcal{A}_{[t-\infty, a]} \lor [b, + \infty]](\omega) = \pi_{[a,b]}(\omega, A), \quad \mu-a.e., \forall A \subset \mathcal{A}_{[a,b]}.
$$
The set $\mathcal{G}(\pi)$ of all local Markov fields specified by $\pi$ is a convex set which is possibly empty.

A local Markov field $\mu \in \mathcal{G}(\pi)$ is extremal iff
\[
\bigcap_n \bigvee_{|\ell| > n} \mathcal{G}_\ell = \{\emptyset, \Omega\}, \quad \mu\text{-a.e. (cf. [2])}.
\]

Theorem 1 implies

**Theorem 2.** Let $(\pi_{(a,b)})_{a < b}$ be local Markov specifications. Every extremal point of $\mathcal{G}(\pi)$ has the Markov property.

**Remarks.** (1) When $T = \mathbb{Z}$, $E$ countable, this result implies Theorem 6 of F. Spitzer (cf. [5]).

(2) This result also implies Theorem 3.9 of G. Royer and M. Yor (cf. [4]).

(3) It should be interesting to know whether similar results hold when $T = \mathbb{Z}^d$ or $\mathbb{R}^d$ with $d \geq 2$.

(4) The characterization of Gibbs states of $\mathcal{G}(\pi)$ possessing the Markov property (other than extremal Gibbs states) is given in [1].

**Bibliography**


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