PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 70, Number 2, July 1978

APPROXIMATION OF INDUCED AUTOMORPHISMS AND SPECIAL AUTOMORPHISMS

P. N. WHITMAN

ABSTRACT. A class of measure-preserving invertible point transformations which admit approximations is defined. If T is an automorphism which admits an approximation, conditions are given such that an induced automorphism and a special automorphism over T again admit an approximation.

1. Preliminaries. Let (X, F, μ) be a measure-space isomorphic to the unit interval with Lebesgue measure. A measure-preserving invertible point transformation of X is called an automorphism of (X, F, μ).

A finite ordered collection $\xi = \{C_i\colon 1 \leqslant i \leqslant q\}$ of pairwise-disjoint measurable sets in X is called a partition. If the union of members of ξ is X, then ξ is called a partition of X. If $A \in F$, we write $A \leqslant \xi$ if A is a union of members of ξ. If $\eta = \{B_j\colon 1 \leqslant j \leqslant p\}$ is a partition, we write $\eta \leqslant \xi$ if $B_j \leqslant \xi$ for $1 \leqslant j \leqslant p$. If $0 < \beta < 1$ we write $\xi \supset \beta\eta$ if each $B_j \in \eta$ contains a $C_i \in \xi$ with $\mu(C_i) \geqslant \beta\mu(B_j)$.

Let ε_X denote the partition of X into points. A sequence of partitions $\{\xi(n)\}$ converges to the unit partition, written $\xi(n) \to \varepsilon_X$, if for each $A \in F$, $\mu(A \triangle A(\xi(n))) \to 0$ as $n \to \infty$, where $A(\xi(n)) \leqslant \xi(n)$ and is such that $\mu(A \triangle A(\xi(n)))$ is a minimum.

The following definitions are due to Katok and Stepin [4] and Chacon [1], respectively.

DEFINITION 1. An automorphism T is said to admit a cyclic approximation by periodic transformations of the first kind (cyclic a. p. t. I) with speed $f(n)$, where $f(n)$ is a sequence of real numbers decreasing to zero, if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$; and
2. $\sum_{i=1}^{q(n)} \mu(TC_i(n) \triangle C_{i+1}(n)) < f(q(n))$ where $C_{q(n)+1}$ means $C_1(n)$.

DEFINITION 2. An automorphism T is said to admit a simple approximation if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$; and
2. $TC_i(n) = C_{i+1}(n)$, $1 \leqslant i \leqslant q(n) - 1$.

It is known that automorphisms which admit a simple approximation are ergodic and have simple spectrum.

Received by the editors August 4, 1977.
AMS (MOS) subject classifications (1970). Primary 28A65.

2. Induced automorphisms and special automorphisms. Let T be an automorphism of (X, F, μ). Let $A \in F$.

DEFINITION 3. The induced automorphism $T_A: A \to A$ is defined by

$$T_A x = T^k x, \qquad x \in A,$$

where k is the least positive integer such that $T^k x \in A$ (neglecting sets of measure zero).

Denote by $\mathbf{N}$ the set of nonnegative integers, and by $\mathbf{R}^+$ the set of positive real numbers. Let $f: X \to \mathbf{N}$ be an integrable function. Put $B(k, n) = \{(x, n): x \in X, f(x) = k\}$. Put $X(f) = \bigcup_{k \geqslant 0} \bigcup_{n=0}^{k} B(k, n)$ and identify X with the set $\bigcup_{k \geqslant 0} B(k, 0)$. We may regard each set $B(k, n)$, $0 < n \leqslant k$, as a copy of $B(k, 0)$. Consequently, we may extend μ to $X(f)$ and form a normalised measure μ' on $X(f)$ in the obvious way.

DEFINITION 4. Let the transformation T_f on $X(f)$ be defined by

$$T_f(x, n) = (x, n+1), \qquad 0 \leqslant n < f(x),$$
$$T_f(x, f(x)) = (Tx, 0).$$

T_f is called the special automorphism over T built under the function f.

If f is the characteristic function of a set $A \in F$, then the special automorphism T_f is denoted T^A and is called a primitive automorphism over T.

Order the sets $B(k, n)$, $k \geqslant 0$, $0 \leqslant n \leqslant k$, lexicographically.

DEFINITION 5. Let ξ be a partition in X such that every element of ξ is contained in exactly one of the sets $B(k, 0)$ for some k. Then ξ^f is the partition in $X(f)$ consisting of the elements $C \in \xi$, together with, for each $C \in \xi$, where $C \subset B(k, 0)$, a copy of C in each of the sets $B(k, n)$, $0 < n \leqslant k$. The ordering on ξ^f is that inherited from the sets $B(k, n)$, $k \geqslant 0$, $0 \leqslant n \leqslant k$.

It is easily seen that if $\xi(n) \to \varepsilon_X$, then $\xi^f(n) \to \varepsilon_{X(f)}$.

Recall that the measure algebra (F, μ) is a complete metric space with respect to the metric d given by

$$d(A, B) = \mu(A \triangle B), \qquad A, B \in F.$$

Goodson [3] has proved the following two theorems for induced and primitive automorphisms.

THEOREM 1. *Let $T: X \to X$ be an automorphism which admits a simple approximation; then there is a collection of subsets of X, dense in F, such that the automorphisms T^A and T_A on any one of these sets also admit a simple approximation.*

THEOREM 2. *Let $T: X \to X$ admit a simple approximation with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n)$ having $q(n)$ elements, and suppose $A \in F$, with $\mu(A) > 0$, can be approximated by sets $A(n) \subset A$ with $A(n) \leqslant \xi(n)$ and such that $q(n)\mu(A \setminus A(n)) \to 0$ as $n \to \infty$. Then T^A and T_A both admit a simple approximation.*

3. Main results. We show below that if T admits a cyclic a.p.t.I with fast enough speed then both T_A and T^A admit a simple approximation, where A is as in Theorem 2. We require the following lemma.

LEMMA 1. *Let $\{\zeta(n)\}$ be a sequence of partitions with $\zeta(n) \to \varepsilon_X$. Let $\{\xi(n)\}$ be a sequence of partitions and let $\beta(n)$ be an increasing sequence of positive real numbers converging to* 1 *such that $\xi(n) \supset \beta(n)\zeta(n)$, $n \geqslant 1$. Then $\xi(n) \to \varepsilon_X$.*

PROOF. Let $A \in F$ with $\mu(A) > 0$. Since $\zeta(n) \to \varepsilon_X$, there is a sequence of sets $A(\zeta(n)) \leqslant \zeta(n)$ such that $\mu(A \triangle A(\zeta(n))) \to 0$ as $n \to \infty$.

Let $D(n) = \bigcup \{C(n) \in \xi(n)\colon C(n) \subset A(\zeta(n))\}$; then

$$\mu\big(A(\zeta(n)) \triangle D(n)\big) \leqslant (1 - \beta(n))\,\mu\big(\bigcup\{C(n)\colon C(n) \in \xi(n)\}\big) \leqslant 1 - \beta(n).$$

Consequently,

$$\mu(A \triangle D(n)) \leqslant \mu\big(A \triangle A(\zeta(n))\big) + \mu\big(A(\zeta(n) \triangle D(n))\big) \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\mu\big(A \triangle A(\xi(n))\big) \leqslant \mu(A \triangle D(n)) \leqslant 0 \quad \text{as } n \to \infty.$$

The proof of the following result is based on techniques shown in **[2]**.

THEOREM 3. *Let $T\colon X \to X$ be an automorphism admitting a cyclic a.p.t.* I *with speed $f(n) = o(1/n^2)$ with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$. Let $A \in F$ be approximated by sets $A(n) \leqslant \xi(n)$ with $A(n) \subset A$ such that $q(n)\,\mu(A \setminus A(n)) \to 0$ as $n \to \infty$. Then T_A and T^A admit a simple approximation.*

PROOF. For each i, $1 \leqslant i \leqslant q(n)$, we can write $C_i(n) = F_i(n) \cup G_i(n)$, where $TG_i(n) \subset C_{i+1}(n)$ and $TF_i(n) \cap C_{i+1}(n) = \varnothing$, $1 \leqslant i \leqslant q(n)$. $C_{q(n)+1}(n)$ means $C_1(n)$.

Let $C_1'(n) = C_1(n) \cap \bigcap_{i=1}^{q(n)} T^{-i+1}G_i(n)$ and put

$$T^{i-1}C_1'(n) = C_i'(n), \qquad 1 \leqslant i \leqslant q(n).$$

As in **[2]**, it can be shown that

$$\mu(C_1(n) \setminus C_1'(n)) < \tfrac{1}{2} f(q(n)).$$

Thus,

$$\mu(C_1'(n)) > \mu(C_1(n)) - \tfrac{1}{2} f(q(n)) = \mu(C_1(n))\big(1 - o(1/q(n))\big).$$

Consequently, by Lemma 1, $\xi'(n) \to \varepsilon_X$, where $\xi'(n) = \{C_i'(n)\colon 1 \leqslant i \leqslant q(n)\}$, and T admits a simple approximation with respect to the sequence $\{\xi'(n)\}$.

Let $A'(n) = \bigcup \{C_i'(n) \in \xi'(n)\colon C_i'(n) \subset A(n)\}$. Then $A'(n) \subset A$ and

$$\begin{aligned}\mu(A \setminus A'(n)) &\leqslant \mu(A \setminus A(n)) + \mu(A(n) \setminus A'(n)) \\ &\leqslant \mu(A \setminus A(n)) + \tfrac{1}{2} q(n) f(q(n)) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Furthermore,

$$q(n)\,\mu(A \setminus A'(n)) \to 0 \quad \text{as } n \to \infty.$$

Hence by Theorem 2, T_A and T^A admit a simple approximation.

Next, we give results for special automorphisms analogous to Theorems 1 and 2. From our results we are able to obtain a slightly weaker version of Theorem 2 for primitive automorphisms as a corollary.

THEOREM 4. *Let T: $X \to X$ be an automorphism admitting a simple approximation with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$. Let f: $X \to \mathbf{N}$ be integrable. Let $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$. Suppose that for each $C_i(n) \in \xi(n)$, $1 \leqslant i \leqslant q(n)$, there exists $k_i(n) \in \mathbf{N}$ such that*

$$\mu\{x \in C_i(n)\colon f(x) = k_i(n)\} \geqslant (1 - \delta(n))\,\mu(C_i(n))$$

where $\delta(n)$ is a sequence of real numbers with $\delta(n) = o(1/q(n))$. Then T_f admits a simple approximation.

PROOF. Let $B_i(n) = f^{-1}(k_i(n))$. Put

$$D_1(n) = \bigcap_{i=0}^{q(n)-1} T^{-i}(C_{i+1}(n) \cap B_{i+1}(n)).$$

Then

$$\begin{aligned}\mu(D_1(n)) &\geqslant \mu(C_1(n)) - q(n)\delta(n)\,\mu(C_1(n))\\ &= \mu(C_1(n))(1 - q(n)\delta(n)). \qquad (1)\end{aligned}$$

Put $D_i(n) = T^{i-1}D_1(n)$, $1 \leqslant i \leqslant q(n)$, and put $\eta(n) = \{D_i(n)\colon 1 \leqslant i \leqslant q(n)\}$. Then $\eta(n) \supset (1 - q(n)\delta(n))\xi(n)$ by (1), and consequently $\eta(n) \to \varepsilon_X$ by Lemma 1. Hence $\eta^f(n) \to \varepsilon_{X(f)}$ and T_f admits a simple approximation with respect to $\{\eta^f(n)\}$.

In the case where f is the characteristic function of a set A, Theorem 4 has the following weaker form of Goodson's result as a corollary.

COROLLARY 1. *Let T: $X \to X$ be an automorphism which admits a simple approximation with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$. Let A be a measurable set approximated by sets $A(n) \leqslant \xi(n)$, $A(n) \subset A$, in the sense that $q(n)^2\mu(A \setminus A(n)) \to 0$ as $n \to \infty$. Then T^A admits a simple approximation.*

PROOF. Let $f(x) = 1$, $x \in A$, $= 0$, $x \in X \setminus A$. Let $C(n) \in \xi(n)$. If $C(n) \subset A(n)$, then clearly $\mu\{x \in C(n)\colon f(x) = 1\} = \mu(C(n))$. If $C(n) \subset X \setminus A(n)$, then

$$\begin{aligned}\mu\{x \in C(n)\colon f(x) = 0\} &= \mu(C(n)) - \mu(C(n) \cap A)\\ &\geqslant \mu(C(n)) - \mu(A \setminus A(n))\\ &= \mu(C(n)) - o\big(1/q(n)^2\big)\\ &= \mu(C(n))\big(1 - o(1/q(n))\big).\end{aligned}$$

Hence by Theorem 4, $T_f = T^A$ admits a simple approximation.

We now give a density result for special automorphisms which is analogous to that of Theorem 1.

THEOREM 5. *The collection of functions $f\colon X \to \mathbf{N}$ which satisfy the conditions of Theorem 4 is dense in the set of integrable functions $g\colon X \to \mathbf{N}$ with the L_1 topology.*

PROOF. Suppose that $g\colon X \to \mathbf{N}$ is integrable, and takes values $k_1 < k_2 < \dots$. Put $G_i = g^{-1}(k_i)$, $i \geqslant 1$. We assume, without loss of generality, that $\mu(G_i) > 0$, $i \geqslant 1$. Now choose $m \in \mathbf{N}$ sufficiently large so that, given $\varepsilon > 0$,

$$\int_{X \setminus \bigcup_{i=1}^m G_i} g \, d\mu < \frac{\varepsilon}{4}. \tag{2}$$

Since $\xi(n) \to \varepsilon_X$, we can find n such that there are $\xi(n)$-sets $E_i(n) \leqslant \xi(n)$, $1 \leqslant i \leqslant m$, which are pairwise disjoint and $\mu(E_i(n) \triangle G_i) < \varepsilon/2mk_m$, $1 \leqslant i \leqslant m$. Define $f(x) = k_i$, $x \in E_i(n)$, $= 0$, $x \in X \setminus \bigcup_{i=1}^m E_i(n)$. Then

$$\begin{aligned}
\int_X |f - g| \, d\mu &= \int_{\bigcup_{i=1}^m G_i} |f - g| \, d\mu + \int_{X \setminus \bigcup_{i=1}^m G_i} |f - g| \, d\mu \\
&= \sum_{i=1}^m \left(\int_{G_i} |f - g| \, d\mu \right) + \int_{X \setminus \bigcup_{i=1}^m G_i} |f - g| \, d\mu \\
&\leqslant \sum_{i=1}^m \int_{G_i \cap E_i(n)} |f - g| \, d\mu + \int_{G_i \setminus E_i(n)} |f - g| \, d\mu + 2 \int_{X \setminus \bigcup_{i=1}^m G_i} g \, d\mu,
\end{aligned}$$

since $f(x) \leqslant g(x)$ if $x \in X \setminus \bigcup_{i=1}^m G_i$. Thus by (2),

$$\begin{aligned}
\int_X |f - g| \, d\mu &< \sum_{i=1}^m \int_{G_i \setminus E_i(n)} |f - g| \, d\mu + \frac{\varepsilon}{2} \leqslant \sum_{i=1}^m \int_{G_i \setminus E_i(n)} |f - k_i| \, d\mu + \frac{\varepsilon}{2} \\
&< \sum_{i=1}^m k_m \mu(G_i \triangle E_i(n)) + \frac{\varepsilon}{2} < \sum_{i=1}^m \frac{k_m \varepsilon}{2mk_m} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

The result below is analogous to Theorem 3.

THEOREM 6. *Let $T\colon X \to X$ be an automorphism which admits a cyclic a.p.t.* I *with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$, with speed $g(n) = o(1/n^2)$. Let $f\colon X \to \mathbf{N}$ be integrable, and suppose that for each $C_i(n) \in \xi(n)$, $1 \leqslant i \leqslant q(n)$, there exists $k_i(n) \in \mathbf{N}$ such that*

$$\mu\{x \in C_i(n)\colon f(x) = k_i(n)\} \geqslant (1 - \varepsilon(n)) \mu(C_i(n)), \tag{3}$$

where $\varepsilon(n)$ is a sequence of real numbers with $\varepsilon(n) = o(1/q(n))$. Then T_f admits a simple approximation.

PROOF. As in Theorem 3, we can show that there exists a sequence of partitions $\xi'(n) = \{C_i'(n)\colon 1 \leqslant i \leqslant q(n)\}$ with $\xi'(n) \to \varepsilon_x$ and $\mu(C_i(n) \setminus C_i'(n)) < \frac{1}{2} g(q(n))$ such that T admits a simple approximation with respect to the sequence $\{\xi'(n)\}$.

We show that T and $\{\xi'(n)\}$ satisfy the conditions of Theorem 4, from which it will follow that T_f admits a simple approximation.

Let $B_i(n) = f^{-1}(k_i(n))$. Then by (3),

$$\begin{aligned}\mu(C_i'(n) \cap B_i(n)) &= \mu(C_i(n) \cap B_i(n)) - \mu((C_i(n) \setminus C_i'(n)) \cap B_i(n)) \\ &\geqslant (1 - \varepsilon(n))\,\mu(C_i(n)) - \tfrac{1}{2}\, g(q(n)) \\ &\geqslant (1 - \varepsilon(n))\,\mu(C_i'(n)) - \tfrac{1}{2}\, g(q(n)).\end{aligned}$$

Put $\delta(n) = \varepsilon(n) + g(q(n))/2\mu(C_1'(n))$; then $\delta(n) = o(1/q(n))$. Hence,

$$\mu\{x \in C_i'(n): f(x) = k_i(n)\} = \mu(C_i'(n) \cap B_i(n)) > (1 - \delta(n))\,\mu(C_i'(n)),$$

and so by Theorem 4, T_f admits a simple approximation.

The above results give conditions for a special automorphism to admit a simple approximation. We now give conditions for a special automorphism to admit a cyclic a.p.t.I with a preassigned speed.

DEFINITION 6. Let T: $X \to X$ be an automorphism. Then T admits a cyclic a.p.t.I with speed $g(x)$ if T admits a cyclic a.p.t.I with speed $g(n)$, where g is a decreasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$ with $g(x) \to 0$ as $x \to \infty$.

THEOREM 7. *Let T: $X \to X$ be an automorphism which admits a cyclic a.p.t.*I *with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$ with speed $g(x) = o(1/x^k)$. Let f: $X \to \mathbf{N}$ be integrable with $f(C_i(n)) = k_i(n) \in \mathbf{N}$, $1 \leqslant i \leqslant q(n)$, $n \geqslant 1$. Then T_f admits a cyclic a.p.t.*I *with speed $o(1/n^k)$.*

PROOF. Let $\xi^f(n) = \{C_{ij}(n)\colon 1 \leqslant i \leqslant q(n), 0 \leqslant j \leqslant k_i(n)\}$, where $C_{i,0}(n) = C_i(n) \in \xi(n)$. For ease of notation put

$$C_{i,k_i(n)+1}(n) = C_{i+1,0}(n) \quad \text{and} \quad C_{q(n)+1,k_{q(n)}+1}(n) = C_{1,0}(n).$$

Then

$$\begin{aligned}&\sum_{i=1}^{q(n)} \sum_{j=0}^{k_i(n)} \mu'(T_f C_{ij}(n) \triangle C_{i,j+1}(n)) \\ &\qquad = \sum_{i=1}^{q(n)} \mu'(T_f C_{i,k_i(n)}(n) \triangle C_{i+1,0}(n)) \\ &\qquad = \left(1 + \int f\,d\mu\right)^{-1} \sum_{i=1}^{q(n)} \mu(TC_i(n) \triangle C_{i+1}(n)) \\ &\qquad < \left(1 + \int f\,d\mu\right)^{-1} g(q(n)).\end{aligned}$$

Let $p(n)$ be the number of elements in $\xi^f(n)$. Then

$$\begin{aligned}p(n) &= q(n) + \sum_{i=1}^{q(n)} k_i(n) = q(n)\left(1 + \frac{1}{q(n)} \sum_{i=1}^{q(n)} k_i(n)\right) \\ &\leqslant q(n)\left(1 + 2\mu(C_1(n)) \sum_{i=1}^{q(n)} k_i(n)\right), \qquad (4)\end{aligned}$$

for n sufficiently large, since $q(n)\,\mu(C_1(n)) \to 1$ and so $q(n)\,\mu(C_1(n)) \geqslant \frac{1}{2}$ for n sufficiently large.

We may assume without loss of generality that (4) holds for all n. Hence,

$$p(n) \leqslant q(n)\left(1 + 2\mu(C_1(n))\sum_{i=1}^{q(n)} k_i(n)\right) \leqslant q(n)\left(1 + 2\int f\,d\mu\right). \tag{5}$$

Now put $G(n) = g(n\,.\,(1 + 2\int f\,d\mu)^{-1})\,.\,(1 + \int f\,d\mu)^{-1}$. Then,

$$\sum_{i=1}^{q(n)} \sum_{j=0}^{k_i(n)} \mu'(T_f C_{ij}(n) \triangle C_{i,j+1}(n)) < \left(1 + \int f\,d\mu\right)^{-1} g(q(n)) \leqslant G(p(n))$$

by (5)

The conclusion of the theorem is now apparent.

By exactly the same reasoning we also have the following.

THEOREM 8. *Let T: $X \to X$ be an automorphism which admits a cyclic a.p.t.*I *with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n)\colon 1 \leqslant i \leqslant q(n)\}$ with speed $g(x) = \theta/x$, $\theta \in \mathbf{R}^+$. Let f: $X \to \mathbf{N}$ be integrable with $f(C_i(n)) = k_i(n) \in \mathbf{N}$, $1 \leqslant i \leqslant q(n)$, $n \geqslant 1$. Then T_f admits a cyclic a.p.t.*I *with speed $G(n) = 2\theta/n$.*

PROOF. As for Theorem 7, noticing that

$$\left(1 + \int f\,d\mu\right)^{-1} g(q(n)) \leqslant \theta\left(1 + 2\int f\,d\mu\right)\Big/p(n)\left(1 + \int f\,d\mu\right) < 2\theta/p(n).$$

It is known [1] that if T admits a cyclic a.p.t.I with speed θ/n, where $\theta < 1$, then T has simple spectrum.

We do not know if, in general, a sufficient condition for T_f to have simple spectrum is that T has simple spectrum.

BIBLIOGRAPHY

1. R. V. Chacon, *Approximations and spectral multiplicity*, Lecture Notes in Math., vol. 160, Springer-Verlag, Berlin and New York, 1970, pp. 18–27.

2. R. V. Chacon and T. Schwartzbauer, *Commuting point transformations*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete. **11** (1969), 277–287. MR **39** #2939.

3. G. R. Goodson, *Induced automorphisms and simple approximations*, Proc. Amer. Math. Soc. **54** (1976), 141–145. MR **52** #10997.

4. A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Russian Math. Surveys (5) **22** (1967), 77–102. MR **36** #2776.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SUSSEX, FALMER, BRIGHTON, ENGLAND

Current address: University of Witwatersrand, Jan Smuts Avenue, Johannesburg, South Africa