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WITT CLASSES OF INTEGRAL REPRESENTATIONS OF AN ABELIAN 2-GROUP

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ABSTRACT. In this paper the Witt groups of integral representations of an abelian 2-group π, $W_0(\pi; Z)$ and $W_2(\pi; Z)$ are calculated. Invariants are listed which completely determine $W_0(Z_4; Z)$ and $W_2(Z_4; Z)$ and can be extended to the case $\pi = Z_{2^k}$. If π is an elementary abelian 2-group, it is shown that $W_2(\pi; Z) = 0$ and $W_0(\pi; Z[\frac{1}{2}])$ is ring isomorphic to the group ring $W(Z[\frac{1}{2}])(\mathrm{Hom}(\pi, Z_2))$.

In [1], Alexander, Conner, Hamrick and Vick studied the Witt classes of integral representations of an abelian p-group; however, their results focused on the case where p is an odd prime. In this paper, we study the case where $p = 2$.

Our interest in this algebra stems from an interest in the bordism of manifolds with a differentiable action of an abelian 2-group, say π, and the Atiyah-Bott homomorphism ab: $\mathcal{O}_*(\pi) \to W_*(\pi; Z)$ (cf. [3]) provides a very convenient bordism invariant. The algebra which we will develop here provides the really essential information for a study along the lines of [3]. For reasons of length, bordism related results will appear in a subsequent publication.

The groups $W_*(Z_{2^k}; Z)$ are computed in §1, and we show that W_0 has rank $2^{k-1} + 1$ and torsion subgroup isomorphic to Z_2 while W_2 is a free abelian group of rank $2^{k-1} - 1$. Complete invariants for $W_*(Z_4; Z)$ are exhibited in §2; these will be applied to equivariant bordism theory elsewhere. In §3, we show that for an arbitrary finite abelian 2-group π, the rank of $W_k(\pi; Z)$ is equal to

$$\tfrac{1}{2}\left[\mathrm{Order}(\pi)(1 - 1/2^L) + \left(1 + (-1)^{k/2}\right)\mathrm{Order}(\mathrm{Hom}(\pi, Z_2))\right],$$

where $L = \log_2(\mathrm{Order}(\pi)) - \dim \mathrm{Hom}(\pi, Z_2)$. Furthermore the torsion subgroup is isomorphic to the group ring $Z_2(\mathrm{Hom}(\pi, Z_2))$ if $k = 0$ and is trivial if $k = 2$. In the case that π is an elementary abelian 2-group, we establish the fact that $W_k(\pi; Z[\frac{1}{2}])$ is ring isomorphic to the group ring $W(Z[\frac{1}{2}])(\mathrm{Hom}(\pi, Z_2))$ if $k = 0$ and is trivial if $k = 2$.

Although the rank of $W_k(\pi; Z)$ is in general quite large, the decomposition

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techniques of this paper together with the multisignature combine to completely determine its free part. The torsion subgroup is substantially more tractable than one might fear in that it lies in the image of $W_k(e(\pi); Z)$, where $e(\pi)$ is the elementary abelian 2-group with the same number of summands as π, under the monomorphism given by iterating the action of each generator of $e(\pi)$ the necessary number of times to get an action of π.

Finally, I would like to express my gratitude to my colleague Neal Stoltzfus for patiently and most helpfully showing me the application of his work **[9]** to my own and thereby simplifying this paper substantially.

1. In our calculation of $W_*(\pi; Z)$, the following lemma is essential.

LEMMA 1.1 **[1]**. *For any p-group* π, $W_2(\pi, Z) \cong W_2(\pi; Z[1/p])$ *and there is a split short exact sequence*

$$0 \to W_0(\pi; Z) \to W_0(\pi; Z[1/p]) \to W(Z_p) \to 0.$$

We first want to calculate $W_0(Z_2k; Z)$. To do so, let $[V, T]$ represent a given class in $W_0(Z_2k; Z[\frac{1}{2}])$ and set $\Sigma^+_{T^{2^{k-1}}} = \{x \in V | T^{2^{k-1}}(x) = x\}$ and $\Sigma^-_{T^{2^{k-1}}} = \{x \in V | T^{2^{k-1}}(x) = -x\}$. Using the fact that $T^{2^{k-1}}$ acts as an isometry, we get an isomorphism

$$W_0\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right) \to W_0\left(Z_{2^{k-1}}; Z\left[\tfrac{1}{2}\right]\right) \oplus W_0^-\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right),$$

where the second summand consists of those inner product spaces with Z_{2^k}-action of the form $(\Sigma^-_{T^{2^{k-1}}}, T)$. We see immediately that our calculation will submit to an inductive procedure once we have calculated $W_0^-(Z_{2^k}; Z[\frac{1}{2}])$ and that is our next goal. To do this, we observe that, in the usual way, an inner product space over $Z[\frac{1}{2}]$ with isometry T such that $T^{2^{k-1}}(x) = -x$ is a module over $Z[\frac{1}{2}][X]/X^{2^{k-1}} + 1 = D$, which is a $Z[\frac{1}{2}]$ order in the algebraic number field of positive 2^kth roots of unity $Q[\lambda]$ since the 2^kth cyclotomic polynomial is $\phi_{2^k}(X) = X^{2^{k-1}} + 1$. In fact, it is the maximal $Z[\frac{1}{2}]$ order in $Q[\lambda]$. We note that D is invariant under the natural involution induced by complex conjugation on $Q[\lambda]$ and we may consider the Witt group of D-valued Hermitian forms on finitely generated projective D-modules, $H(D)$. Conveniently, we then have the following theorem.

THEOREM 1.2 **[5]**, **[4]**, **[9]**. *There is a natural isomorphism* $W_0^-(Z_{2^k}; Z[\frac{1}{2}]) \cong H(D)$.

PROOF. We first view elements of $W_0^-(Z_{2^k}; Z[\frac{1}{2}])$ as Witt classes of modules over D with inner products taking their values in $Z[\frac{1}{2}]$ in the way suggested above. Then, using the Trace Lemma and the notation in its formulation as Lemma 2.6 of **[9]**, we let $R = L = Z[\frac{1}{2}]$, $A = K = D$ and s: $K \to L$ be given by $s(\Sigma_{i=0}^{2^{k-1}-1}\alpha_i\lambda^i) = 2^{k-1}\alpha_0$. All the conditions of the Trace Lemma are now rather easily seen to be satisfied and our proof is complete.

It will be immediately useful for us to know that the Dedekind domain D is unramified over $Z[\frac{1}{2}]$. However, using a classical theorem of Euler (cf. **[8]**)

and the fact that 2 is invertible,

$$\Delta^{-1}\left(D/Z\left[\tfrac{1}{2}\right]\right) = \frac{1}{\phi'_{2^k}(\lambda)} D = \frac{1}{2^{k-1}(\lambda)^{2^{k-1}-1}} D = D.$$

Here,

$$\Delta^{-1}\left(D/Z\left[\tfrac{1}{2}\right]\right) = \left\{e \in Q[\lambda] | \operatorname{trace}_{Q[\lambda]/Q}(eD) \subseteq Z\left[\tfrac{1}{2}\right]\right\}$$

is the inverse different [9] (complementary set to the trace in Lang's *Algebraic number theory*). That D is unramified over $Z[\frac{1}{2}]$ follows from the fact that any ramified prime must divide the inverse different. In particular, D is unramified over the fixed ring of the involution.

Now each conjugate pair of embeddings of $Q[\lambda]$ in the complex numbers which preserves the involution yields a signature homomorphism σ: $H(D) \to H(C) \cong Z$. Because $\phi_{2^k}(X) = X^{2^{k-1}} + 1$ has $m = 2^{k-2}$ distinct conjugate pairs of complex roots, there are 2^{k-2} such embeddings and thus the same number of signatures. Since any signature is congruent to the rank modulo 2, the homomorphism $r_i(n_1, \ldots, n_m) = n_i - n_1$ is trivial modulo 2 on the image of the signature homomorphism (σ_i): $H(D) \to Z^m$.

THEOREM 1.3. *$H(D)$ is torsion free and determined by the signature invariant. The possible signatures are given by the exact sequence*

$$0 \to H(D) \xrightarrow{(\sigma_i)} Z^m \xrightarrow{(r_i)} (Z_2)^{m-1} \to 0.$$

PROOF [9, THEOREM 4.11]. Note that $\Delta^{-1}(D/Z[\frac{1}{2}]) = D$ because there are no primes ramified over the fixed ring of D. Now, by Proposition 4.7 of [9], $H(D) = H(\Delta^{-1}(D/Z[\frac{1}{2}])) = \bigcap_m$ Kernel ∂_m, the intersection in the last term being taken over all involution invariant nondyadic maximal ideals. There is exactly one dyadic prime ideal, $(1 - \lambda)$, in the ring of integers $R \subset Q[\lambda]$. By Proposition 4.9 of [9] any even dimensional form in $H(D)$ must have trivial Hilbert symbol at all inert and split primes except $(1 - \lambda)$. Let J be the ideal of even forms and d: $H(D) \to Z_2$ be the discriminant at the dyadic prime $(1 - \lambda)$. Then there is an exact sequence

$$0 \to J \cap H(D) \xrightarrow{(\sigma_i, d)} (2Z)^m \oplus Z_2 \xrightarrow{H} \dot{Z} = \{\pm 1\} \to 0,$$

where H is the Hilbert reciprocity map. This exact sequence yields the fact that $H(D)$ has elements with prescribed even signatures. Since a rank one form $\langle \lambda + \lambda^{-1} \rangle$ exists (the norm from $Q[\lambda]$ to Q of $\lambda + \lambda^{-1}$ is a power of 2 and hence a unit in D), the theorem follows.

We have proved that $W_0(Z_{2^k}; Z[\frac{1}{2}])$ is isomorphic to $W_0(Z_{2^{k-1}}; Z[\frac{1}{2}]) \oplus (Z)^{2^{k-2}}$. Now using the fact (cf. [3]) that $W_0(Z_2; Z[\frac{1}{2}]) \cong Z \oplus Z \oplus Z_2 \oplus Z_2$ and Lemma 1.1, we get the next theorem.

THEOREM 1.4. $W_0(Z_{2^k}; Z) \cong (Z)^{2^{k-1}+1} \oplus Z_2$.

En route to calculating $W_2(Z_{2^k}; Z)$, we note that there is an isomorphism

$$W_2\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right) \to W_2\left(Z_{2^{k-1}}; Z\left[\tfrac{1}{2}\right]\right) \oplus W_2^-\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right),$$

which is defined in exactly the same way as (1) above. Given $(\Sigma^-_{T^{2^{k-1}}}, T)$ with symplectic inner product (,), we define a symmetric inner product

$$\langle x, y\rangle = \left(x, T^{2^{k-2}} y\right), \quad \text{where } x, y \in \Sigma^-_{T^{2^{k-1}}},$$

and thus a homomorphism

$$\mathcal{S}: W_2^-\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right) \to W_0^-\left(Z_{2^k}; Z\left[\tfrac{1}{2}\right]\right).$$

We observe that we can recover the original symplectic inner product by simply verifying that $(x, y) = \langle x, -T^{2^{k-2}} y\rangle$, where $x, y \in \Sigma^-_{T^{2^{k-1}}}$. In fact, this method of switching back and forth between symplectic and symmetric inner product spaces gives us an isomorphism between $W_2^-(Z_{2^k}; Z[\frac{1}{2}])$ and $W_0^-(Z_{2^k}; Z[\frac{1}{2}])$. This, together with the relatively easy observation (first made by P. E. Conner) that $W_2(Z_2; Z) = 0$ and Lemma 1.1, yields the following theorem.

THEOREM 1.5. *$W_2(Z_{2^k}; Z)$ is a free abelian group of rank $2^{k-1} - 1$.*

2. This section is devoted to producing invariants which completely determine $W_*(Z_4; Z)$. As we noted in the introduction, the use of the multisignature and appropriate decompositions will take care of determining the free abelian part of $W_*(\pi; Z)$ for π an abelian 2-group.

The case of $W_2(Z_4; Z)$ is the easier of the two cases under consideration. Here we simply note that the composite homomorphism

$$\begin{aligned} W_2(Z_4; Z) &\to W_2\left(Z_4; Z\left[\tfrac{1}{2}\right]\right) \to W_2^-\left(Z_4; Z\left[\tfrac{1}{2}\right]\right) \\ &\xrightarrow{\mathcal{S}} W_0^-\left(Z_4; Z\left[\tfrac{1}{2}\right]\right)^{1/2} \xrightarrow{\text{sgn}} Z \end{aligned} \tag{$*$}$$

is an isomorphism and sum up in the following theorem.

THEOREM 2.1. *$W_2(Z_4; Z)$ is completely determined by the isomorphism $(*)$.*

We recall from §1 that $W_0(Z_4; Z[\frac{1}{2}])$ is isomorphic to $W_0(Z_2; Z[\frac{1}{2}]) \oplus W_0^-(Z_4; Z[\frac{1}{2}])$ and take note of the fact (arising in part from our calculations in §1) that $W_0(Z_2; Z)$ is mapped monomorphically onto a direct summand of $W_0(Z_{2^k}; Z)$ by simply iterating the action of the generator of Z_2 to get an action of Z_{2^k}. Following the lead now of the above and recalling Proposition 3 and Theorem 10 of [3], our next result follows:

THEOREM 2.2. *$W_0(Z_4; Z)$ is completely determined by the invariants*
(i) $\frac{1}{2}$ sgn: $W_0^-(Z_4; Z[\frac{1}{2}]) \to Z$,
(ii) $\text{sgn}\langle x, y\rangle | \Sigma^+_{T^2}$,
(iii) $\text{sgn}\langle x, Ty\rangle | \Sigma^+_{T^2}$, *and*
(iv) trs, *where the invariant* trs *is given by taking two Reiner decompositions of the integral form, the first with respect to T^2 and the second is of F_{T^2} with respect to T. In the second decomposition, the number of copies of $Z(Z_2)$ is congruent to* trs *modulo* 2.

Before proceding, we want to indicate how the invariant trs should be defined for $W_0(Z_4; Z)$. Namely, it is the image of $[\Sigma_T^+]$, or equivalently $[\Sigma_T^-]$, under the boundary homomorphism $\partial\colon W(Z[\frac{1}{2}]) \to W(Z_2) \cong Z_2$. We should further point out that Theorem 2.3 is easily extended to give invariants completely determining $W_0(Z_{2^k}; Z)$ for any positive integer k. To do so, one merely has to decompose $W_0(Z_{2^k}; Z)$ in the manner performed in §1, use the multisignature and, finally, the obvious version of trs.

3. We now employ the foregoing techniques to garner some information about $W_*(\pi; Z[\frac{1}{2}])$, where π is an abelian 2-group. Of course, Lemma 1.1 reassures us that we can translate this into knowledge of $W_*(\pi; Z)$ at will.

THEOREM 3.1. *Let π be an elementary abelian 2-group. Then $W_2(\pi; Z) = 0$ and $W_0(\pi; Z[\frac{1}{2}])$ is ring isomorphic to the group ring $W(Z[\frac{1}{2}])(\mathrm{Hom}(\pi, Z_2))$.*

PROOF. To see that $W_2(\pi; Z) = 0$, we first view π as $\langle T_1\rangle \oplus \cdots \oplus \langle T_n\rangle$ with T_i^2 equal to the identity. Let $[V, \pi]$ represent an element of $W_2(\pi; Z[\frac{1}{2}])$. Decompose V into $\Sigma_{T_1}^+$ and $\Sigma_{T_1}^-$ and note that $\Sigma_{T_1}^+$ and $\Sigma_{T_1}^-$ are invariant under $T_2, \ldots, T_n$ because π is abelian. Now decompose $\Sigma_{T_1}^+$ and $\Sigma_{T_1}^-$ with respect to T_2 and continue to decompose the result with respect to $T_3, T_4, \ldots, T_{n-2}$ and finally T_{n-1}. In this way we have decomposed $[V, \pi]$ into a direct sum of elements of $W_2(Z_2; Z[\frac{1}{2}])$ on which T_i, $1 \leqslant i \leqslant n-1$, acts as multiplication by ± 1. Because $W_2(Z_2; Z[\frac{1}{2}]) = 0$, each such element of $W_2(Z_2; Z[\frac{1}{2}])$ has a splitter (a T_n-invariant submodule which is equal to its own annihilator) and the direct sum of all these splitters provides us with a splitter for $[V, \pi]$. This concludes the proof that $W_2(\pi; Z[\frac{1}{2}])$ and, hence, $W_2(\pi; Z)$ equals zero. The proof that $W_0(\pi; Z[\frac{1}{2}]) \cong W(Z[\frac{1}{2}])(\mathrm{Hom}(\pi, Z_2))$ involves the same kind of decomposition. The isomorphism is given, after the decomposition has been made with respect to $T_1, \ldots, T_n$, by taking a typical summand, $W(Z[\frac{1}{2}])$, of $W_0(\pi; Z[\frac{1}{2}])$ corresponding to a certain isotropy subgroup of π and associating this summand with the element of $\mathrm{Hom}(\pi, Z_2)$ which maps T_i into zero if and only if T_i is in the isotropy subgroup. It is now a straightforward exercise to prove that this is indeed a ring isomorphism and we omit the details.

Our remaining results on abelian 2-groups are obtained by simply pushing the techniques of §1 and this section a bit further in a rather obvious way. So let π be an abelian 2-group, which we choose to write as a direct sum of cyclic subgroups of nonincreasing orders as read from left to right. Writing $\pi \cong Z_{2^k} \oplus Z_{2^n} \oplus \pi'$, we observe as before that

$$W_0\left(\pi; Z\left[\tfrac{1}{2}\right]\right) \cong W_0^-\left(Z_{2^k} \oplus Z_{2^n} \oplus \pi'; Z\left[\tfrac{1}{2}\right]\right) \oplus W_0\left(Z_{2^{k-1}} \oplus Z_{2^n} \oplus \pi'; Z\left[\tfrac{1}{2}\right]\right), \tag{2}$$

where the first summand is that on which the generator T of Z_{2^k} iterated 2^{k-1} times, $T^{2^{k-1}}$, acts as multiplication by -1. We now concentrate on $W_0^-(Z_{2^k} \oplus Z_{2^n} \oplus \pi'; Z[\frac{1}{2}])$. Just as in §1 we were able to view $W_0^-(Z_{2^k}; Z[\frac{1}{2}])$ as $W(D)$ subject to the stipulation that the inner products take their values in $Z[\frac{1}{2}]$, we can as well view $W_0^-(Z_{2^k} \oplus Z_{2^n} \oplus \pi'; Z[\frac{1}{2}])$ as $W_0(Z_{2^n} \oplus \pi'; D)$, where $Z_{2^n} \oplus \pi'$ acts as a group of D-module isometries and the inner products continue to take all their values in $Z[\frac{1}{2}]$. The advantage of this construction is that we now can break up the action of Z_{2^n} on a given D-module into an orthogonal direct sum of D-modules on which Z_{2^n} acts as multiplication by the various 2^nth roots of unity. Repeating this procedure, we arrive at the following lemma (with the aid of Theorem 1.3).

LEMMA 3.2. *$W_0^-(Z_{2^k} \oplus Z_{2^n} \oplus \pi'; Z[\frac{1}{2}])$ is a free abelian group of rank* $\frac{1}{4}$ Order(π).

Rewriting the order in which the summands occur as it becomes necessary in order to continue having cyclic subgroups with nonincreasing orders, we now work on the right-hand summand of (2) and deduce by induction that $W_0(\pi; Z[\frac{1}{2}])$ is isomorphic to the direct sum of $W_0(e(\pi); Z[\frac{1}{2}])$ and a free abelian group of rank $\frac{1}{4}$ Order$(\pi)[1 + \frac{1}{2} + \cdots + 1/2^{L-1}]$, where $e(\pi)$ is the elementary abelian 2-group with the same number of cyclic summands as π and $L = \log_2[\text{Order}(\pi)] - \dim \text{Hom}_{Z_2}(\pi, Z_2)$. This gives us our next theorems.

THEOREM 3.3. *$W_0(\pi; Z[\frac{1}{2}])$ is isomorphic to the direct sum of $W(Z[\frac{1}{2}])(\text{Hom}(\pi, Z_2))$ and a free abelian group of rank* $\frac{1}{2}[1 - 1/2^L]$ Order(π), *where* $L = \log_2[\text{Order}(\pi)] - \dim \text{Hom}_{Z_2}(\pi, Z_2)$.

THEOREM 3.4. *$W_2(\pi; Z[\frac{1}{2}])$ is a free abelian group of rank* $\frac{1}{2}[1 - 1/2^L]$Order(π), *where* $L = \log_2[\text{Order}(\pi)] - \dim \text{Hom}_{Z_2}(\pi, Z_2)$.

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