A REFINEMENT OF THE ARITHMETIC MEAN-
GEOMETRIC MEAN INEQUALITY

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Abstract. Upper and lower bounds are given for the difference between the
arithmetic and geometric means of \( n \) positive real numbers in terms of the
variance of these numbers.

In this note we prove a simple refinement of the arithmetic mean-geometric
mean inequality. Our result solves a problem posed by Kenneth S. Williams
in [5] and generalizes an inequality on p. 215 of [3]. Other estimates for the
difference between the means are discussed in [2], [3] and [4].

Theorem. Suppose that \( x_k \in [a, b] \) and \( p_k > 0 \) for \( k = 1, \ldots, n \), where
\( a > 0 \), and suppose that \( \sum_{k=1}^{n} p_k = 1 \). Then, writing \( \bar{x} = \sum_{k=1}^{n} p_k x_k \), we have

\[
\frac{1}{2b} \sum_{k=1}^{n} p_k (x_k - \bar{x})^2 \leq \bar{x} - \left( \prod_{k=1}^{n} x_k^{p_k} \right) \leq \frac{1}{2a} \sum_{k=1}^{n} p_k (x_k - \bar{x})^2. \tag{1}
\]

In particular, if \( p_k = 1/n \) for each \( k \), then

\[
\frac{1}{2bn^2} \sum_{j<k} (x_j - x_k) \leq \frac{x_1 + \cdots + x_n}{n} - \left( \prod_{j=1}^{n} x_j \right)^{1/n} \leq \frac{1}{2an^2} \sum_{j<k} (x_j - x_k)^2.
\]

Remark. These inequalities may be generalized as follows: Let \( m \) be a
probability measure on \([a, b]\), where \( a > 0 \), and let \( \mu = \int_a^b t \, dm(t) \) and
\( \sigma^2 = \int_a^b (t - \mu)^2 \, dm(t) \) be the mean and variance of \( m \). Then

\[
\frac{1}{2b} \sigma^2 \leq \mu - \exp \left( \int_a^b \log(t) \, dm(t) \right) \leq \frac{1}{2a} \sigma^2.
\]

This follows from our theorem and the weak* density of the measures of the form \( \sum_{k=1}^{n} p_k \delta_{x_k} \) (where \( \delta_x \) denotes the probability measure which is
concentrated at the point \( x \)) in the set of all probability measures on \([a, b]\).
(See [1, p. 709].) Notice that the inequality

\[
\exp \left( \int_a^b \log(t) \, dm(t) \right) \leq \mu
\]
Lemma. Let $0 < q < 1$. Then for all $t > 0$ we have

$$1 + qt + \frac{q(q - 1)}{2} t^2 < (1 + t)^q < 1 + qt + \frac{q(q - 1)}{2} \frac{t^2}{1 + t}.$$ 

Proof. After a little algebra we see that

$$\frac{d}{dt} \log \left( 1 + qt + \frac{q(q - 1)}{2} \frac{t^2}{1 + t} \right) = \frac{q}{1 + t} \left\{ \frac{2 + (2 + 2q)t + (1 + q)t^2}{2 + (2 + 2q)t + q(1 + q)t^2} \right\}$$

$$> \frac{q}{1 + t} \text{ since } 0 < q < 1$$

$$= \frac{d}{dt} \log(1 + t)^q.$$ 

Since $(1 + t)^q$ and $1 + qt + (q(q - 1)/2)(t^2/(1 + t))$ agree at $t = 0$, the right-hand inequality is proved.

The left-hand inequality may be proved in the same way, or by using the Taylor expansion of $(1 + t)^q$.

Proof of the theorem. The inequalities (1) are trivially valid if $n = 1$. Let $n = 2$. We may suppose that $x_2 > x_1$. Writing $x_2 = (1 + t)x_1$, with $t > 0$, and writing $p_2 = q$, $p_1 = 1 - q$, the desired inequalities (1) become

$$\frac{q(1 - q)}{2b} t^2 x_1^2 < x_1 \left( 1 + qt - (1 + t)^q \right) < \frac{q(1 - q)}{2a} t^2 x_1^2,$$

which follows immediately from our lemma, noting that $a < x_1 < (1 + t)x_1 < b$.

Suppose now that $n > 3$ and that the inequalities (1) have been proved for all admissible $x_k$'s and $p_k$'s with $n - 1$ replacing $n$.

Fix $x_1, \ldots, x_n$. We may assume that the $x_k$'s are distinct, for otherwise the inequalities follow from the induction hypothesis. Let us consider the left-hand inequality. Define

$$f(p) = f(p_1, \ldots, p_n) = \bar{x} - \prod_{k=1}^n (x_k^p) - \frac{1}{2b} \sum_{k=1}^n p_k (x_k - \bar{x})^2$$

for $p \in S = \{ p = (p_1, \ldots, p_n); p_k > 0 \text{ for each } k \}$.

There is a point $p^o$ of $S$ where $f$ is minimized subject to the constraint $\sum p_k = 1$. If $p^o$ lies on the boundary of $S$, then some component of $p^o$ is zero, and hence $f(p^o) > 0$ by the induction hypothesis, and so the left-hand inequality holds.

If $p^o$ is an interior point of $S$, then we may use the Lagrange multiplier method to obtain a real number $\lambda$ such that at $p^o$,

$$\frac{\partial f}{\partial p_j} = \lambda \frac{\partial}{\partial p_j} \left( \sum_{k=1}^n p_k - 1 \right) \text{ for all } j.$$
i.e.

\[ x_j - (\log x_j) \prod_{1}^{n} (x_k^b) - \frac{(x_j - \bar{x})^2}{2b} = \lambda. \]

Thus each \( x_j \) is a solution of the equation (in \( \xi \))

\[(1 + \frac{\bar{x}}{b})\xi - \bar{x} \log (\xi) - \xi^2 / 2b = \lambda + \bar{x}^2 / 2b \tag{2}\]

(writing \( \bar{x} \) for \( \prod(x_k^b) \)).

Now between any two roots of (2) there is by Rolle’s theorem a root of

\[ 1 + \frac{\bar{x}}{b} - \frac{\bar{x}}{\xi} - \frac{\xi}{b} = 0, \]

i.e. of

\[ \xi^2 - (b + \bar{x})\xi + b\bar{x} = 0. \tag{3} \]

Since (3) has at most 2 solutions, equation (2) has at most 3 solutions. The larger root of (3) is, since \( \bar{x} < \bar{x} \),

\[ \left( b + \bar{x} + \sqrt{(b + \bar{x})^2 - 4b\bar{x}} \right) / 2 > b. \]

Hence equation (2) has at most 2 solutions in \([a, b] \). Since each \( x_j \) is a solution and since the \( x_j \)'s are distinct, we must have \( n < 2 \), contrary to assumption.

Thus \( p^o \) must be a boundary point of \( S \), and so the left-hand inequality is proved.

The right-hand inequality may be proved in the same way by replacing \( b \) by \( a \) in the definition of \( f \) and by noting that the smaller root of the equation corresponding to (3) is \( < a \).

**Remark.** Examination of the above proof shows that the inequalities in (1) are strict unless the \( x_k \)'s corresponding to nonzero \( p_k \)'s are all equal. Furthermore, the constants \( 1/2a \) and \( 1/2b \) in (1) are the best possible. For in the case \( n = 2 \) we have

\[ \frac{\bar{x} - \prod(x_k^a)}{\sum p_k (x_k - \bar{x})^2} = \frac{1 + qt - (1 + t)^q}{q(1 - q)^2x_1} \]

if \( 0 < q < 1 \) and \( t > 0 \) (in the notation of the first paragraph of the proof). It is easy to see that the limit of this expression as \( t \) tends to zero is \( 1/2x_1 \), and since \( x_1 \in [a, b] \) is arbitrary, the result follows.

**References**