ON THE CONVERGENCE OF SOME ITERATION PROCESSES IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. For the approximation of fixed points of a nonexpansive operator $T$ in a uniformly convex Banach space $E$ the convergence of the Mann-Toeplitz iteration $x_{n+1} = \alpha_n T(x_n) + (1 - \alpha_n)x_n$ is studied. Strong convergence is established for a special class of operators $T$. Via regularization this result can be used for general nonexpansive operators, if $E$ possesses a weakly sequentially continuous duality mapping. Furthermore strongly convergent combined regularization-iteration methods are presented.

Throughout this note, let $(E, |\cdot|)$ be a uniformly convex Banach space, let $C$ be a nonempty closed convex subset of $E$. Let $T: C \to C$ denote a nonexpansive operator, i.e. $|T(x) - T(y)| \leq |x - y|$ holds for all $x, y \in C$. To approximate a fixed point of $T$ we define the following iterative method (Mann-Toeplitz process) by

$$x_{n+1} = \alpha_n T(x_n) + (1 - \alpha_n)x_n, \quad \alpha_n \in [0, 1] \quad (n > 1).$$

(1)

We make the assumptions that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\alpha_n \in [0, b]$ with $b \in (0, 1)$ for almost all positive integers $n$.

Let us recall that any mapping $J: E \to E^*$ which fulfills

$$(J(u), u) = |J(u)| \cdot |u|, \quad |J(u)| = |u|$$

for all $u \in E$ is termed a duality mapping. We verify easily (see also [3, Theorem 8.9]) that the nonexpansive operator $T$ satisfies

$$(x - y - T(x) + T(y), J(x - y)) \geq 0$$

for all $x, y \in C$. This means that the operator $S := I - T$ is accretive. Now we call an operator $S: C \to E \varphi$-accretive if there exists a function $\varphi: [0, \infty) \to [0, \infty)$ strictly increasing with $\varphi(0) = 0$, and $\lim_{t \to \infty} \varphi(t) = \infty$ such that (cf. [1] for monotone operators)

$$(S(x) - S(y), J(x - y)) \geq \varphi(|x|) - \varphi(|y|) \cdot [ |x| - |y| ]$$

$\forall x, y \in C$. (2)

If $S$ satisfies the stronger condition

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(S(x) - S(y), J(x - y)) \geq \varphi(|x - y|) \cdot |x - y| \quad \forall x, y \in C, \quad (3)

then S is called uniformly \(\varphi\)-accretive.

**Theorem 1.** Let the fixed point set \(F\) of the operator \(T\) be nonempty. Suppose the operator \(S = I - T\) is \(\varphi\)-accretive. Then the Mann-Toeplitz sequence \(\{x_n\}\) converges strongly to the unique fixed point \(p \in F\).

**Proof.** Since for any fixed \(p \in F\)
\[
|x_{n+1} - p| \leq |x_n - p|,
\]
the sequence \(\{x_n\}\) is bounded. Now assume \(p_1, p_2\) belong to \(F\). By (2) it follows that
\[
\left[ \varphi(|p_1|) - \varphi(|p_2|) \right] \cdot [|p_1| - |p_2|] = 0,
\]
therefore \(|p_1| = |p_2|\). On the other hand \(F\) is convex, and is contained in the uniformly convex space \(E\). So \(F\) reduces to a single point \(p\).

According to a result of Ishikawa [7, Lemma 2] \(S(x_n)\) converges strongly to zero, and by a theorem due to Browder [3, Theorem 8.4, p. 103] the sequence \(\{x_n\}\) converges weakly to the unique fixed point \(p\). Since \(J\) is a bounded operator, the sequence \(\{J(x_n - p)\}\) remains bounded. Hence (2) implies that
\[
\left[ \varphi(|x_n|) - \varphi(|p|) \right] \cdot [|x_n| - |p|] \to 0 \quad (n \to \infty).
\]

It follows easily (cf. [1, p. 61]) that \(|x_n| \to |p|\). This yields the claimed norm convergence of the sequence \(\{x_n\}\) in the uniformly convex space \(E\).

Since strictly contractive operators are uniformly \(\varphi\)-accretive, Theorem 3 contains a result in [5, Theorem 1]. Let us note that in a Hilbert space \(E\) the gradient \(f'\) of a Gateaux differentiable, convex functional \(f\) is \(\varphi\)-accretive, if
\[
f((x + y)/2) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y) - \left[ \varphi(|x|) - \varphi(|y|) \right] : [|x| - |y|]
\]
is valid for all \(x, y \in C\). This fact follows from the estimate
\[
(f'(x) - f'(y), x - y) \geq 2 \left[ f(x) + f(y) - 2f((x + y)/2) \right].
\]

Furthermore Theorem 1 remains true, if the inequality (2) is only assumed to hold for all \(x \in C\) and all \(p \in F\), i.e. if
\[
(x - T(x), J(x - p)) \geq \left[ \varphi(|x|) - \varphi(|p|) \right] \cdot [|x| - |p|] \quad \forall x \in C, \forall p \in F
\]
is assumed. Operators \(T: C \to C\) that satisfy for any \(x, y \in C\)
\[
|T(x) - T(y)| \leq a_1|x - T(x)| + a_2|y - T(y)| + a_3|x - y| + a_4|x - T(y)| + a_5|y - T(x)|
\]
with \(a_i > 0 (i = 1, \ldots, 5)\) and \(\sum_{i=1}^{5} a_i < 1\) belong to this class, provided
\[
2a_1 + a_3 + a_4 + a_5 < 1
\]
holds.

Even Theorem 1 can be applied to general nonexpansive operators \(T_\varepsilon\), for then the operators \(S_\varepsilon = (1 - \varepsilon)(I - T) + \varepsilon R (\varepsilon > 0)\) inherit the (uniform) \(\varphi\)-accretiveness from the "regularization operator" \(R\). This observation motivates the following study of the regularization method involved.
Theorem 2. Let the fixed point set $F$ of $T$ in $C$ be nonempty. Let $R: C \rightarrow E$ be a continuous, bounded operator. Suppose $R$ is uniformly $\varphi$-accretive with respect to an odd, weakly sequentially continuous duality mapping $J: E \rightarrow E^*$. Choose positive reals $\delta_k$, and $\varepsilon_k \in (0, 1)$ with $\lim_{k \rightarrow \infty} \delta_k = 0$, and $\lim_{k \rightarrow \infty} \delta_k \varepsilon_k^{-1} = 0$. If the approximate solutions $y_k \in C$ satisfy

$$(1 - \varepsilon_k)(I - T)(y_k) + \varepsilon_k R(y_k) \leq \delta_k,$$

then the sequence $\{y_k\}$ converges strongly to a fixed point $\hat{p}$, which is uniquely determined by the variational inequality

$$(R(\hat{p}), J(\hat{p} - p)) < 0 \quad \forall p \in F. \quad (4)$$

Proof. Let $p \in F$, and set

$$\beta_k = ((1 - \varepsilon_k)(I - T)(y_k) + \varepsilon_k R(y_k), J(y_k - p)).$$

We notice that $|\beta_k| < |y_k - p| \delta_k$. Since $I - T$ is accretive, it follows

$$(R(y_k), J(y_k - p)) \leq \beta_k \varepsilon_k^{-1}. \quad (5)$$

Let us prove the boundedness of the sequence $\{y_k\}$. On account of (5), (3) we conclude

$$\beta_k \varepsilon_k^{-1} + |R(p)| \cdot |y_k - p| \geq \beta_k \varepsilon_k^{-1} - (R(p), J(y_k - p))$$
$$\geq (R(y_k) - R(p), J(y_k - p))$$
$$\geq \varphi(|y_k - p|) \cdot |y_k - p|.$$

We may assume without loss of generality that $|y_k - p|$ is positive, and hence we obtain

$$\delta_k \varepsilon_k^{-1} + |R(p)| \geq \varphi(|y_k - p|).$$

The boundedness of $\{y_k\}$ is immediate, and with a constant $c_p$, dependent only on $p \in F$, (5) reads

$$(R(y_k), J(y_k - p)) \leq c_p \delta_k \varepsilon_k^{-1}. \quad (6)$$

As $R$ is a bounded operator, $\varepsilon_k R(y_k)$ converges to zero. Since the nonexpansive operator $T$ is also bounded, and $\delta_k \rightarrow 0$, we conclude that $(I - T)(y_k)$ converges strongly to zero. By a theorem due to Browder [3, Theorem 8.4] all weak limit points of $\{y_k\}$, which exist by the boundedness of $\{y_k\}$, belong to $F$. Let $\tilde{y} = \text{w-lim}_{i \rightarrow \infty} y_k$; then (3) and (6) imply that

$$c_p \delta_k \varepsilon_k^{-1} - (R(\tilde{y}), J(\tilde{y}_k - \tilde{y})) \geq \varphi(|\tilde{y}_k - \tilde{y}|) \cdot |\tilde{y}_k - \tilde{y}|.$$

Since $J$ is weakly sequentially continuous, we see at once that $\tilde{y} = \lim_{i \rightarrow \infty} y_k$, and (6) results in the claimed inequality (4). Finally we have to show that this inequality uniquely determines $\hat{p} \in F$, thus proving the convergence of the entire sequence $\{y_k\}$. Fix some $p_1, p_2 \in F$ that satisfy (4), then

$$(R(p_1), J(p_1 - p_2)) < 0, \quad -(R(p_2), J(p_1 - p_2)) < 0.$$

The summation of both these inequalities yields $p_1 = p_2$, since $R$ is uniformly $\varphi$-accretive.

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If the regularization operator $R$ is only $\varphi$-accretive, then similar but more involved arguments show that the sequence $\{\tilde{y}_k\}$ is bounded, every weak limit point of $\{\tilde{y}_k\}$ is also a strong limit point, and every limit point belongs to the fixed point set and satisfies (4). But as $J$ is not linear, unless $E$ is a Hilbert space, the set of points which fulfill (4) is generally not convex; therefore uniqueness cannot be obtained as in the proof of Theorem 1.

The approximate solutions $y_k$ can be constructed by finitely many Mann-Toeplitz iterations for the operator $T_k = (1 - \varepsilon_k)T + \varepsilon_k(I - R)$ by Theorem 1, provided $I - R: C \to C$ is nonexpansive. If furthermore $C$ is bounded, fixed points of $T$ and of each $T_k$ exist.

The simplest regularization method is given by $R(x) = x - x^0$, $x^0$ fixed in $C$. In this case Reich [10, Corollary] has already established the strong convergence of the exact solutions $y_k = (1 - \varepsilon_k)T(y_k) + \varepsilon_kx^0$ ($\delta_k = 0$) to a fixed point of $T$ under similar conditions. In view of the inequality (4) which is achieved by regularization, other choices of $R$ should be taken into consideration.

Inspired by the work of Brück [4], and Halpern [6] we combine in conclusion Mann-Toeplitz iteration and regularization to the following iteration process

$$z_1 \in C, \quad z_{m+1} = \alpha_m(1 - \varepsilon_m)T(z_m) + \alpha_m\varepsilon_mU(z_m) + (1 - \alpha_m)z_m. \quad (7)$$

Here we require that $U: C \to C$ is a strict contraction with contraction constant $q \in [0, 1)$. Clearly $R = I - U$ is then uniformly $\varphi$-accretive with $\varphi(t) = (1 - q)t$. Let us note that the choice $U(z) = \tilde{z}, \tilde{z}$ fixed in $C$, reduces (7) with $\varepsilon_m = \Theta_m(1 + \Theta_m)^{-1}$, $\alpha_m = \lambda_m(1 + \Theta_m)$ to the iteration method which is considered in [4, p. 123], and is also contained in the projection-iteration method of Bakusinskii and Poljak [2, Theorem 3D] for the solution of variational inequalities in Hilbert spaces.

The subsequent results hold in arbitrary Banach spaces $E$.

**Theorem 3.** Let $C$ be a bounded closed convex subset of $E$. Suppose the sequence $\{y_i\}$ converges to a fixed point $p$ of $T$, where $y_i$ is given by

$$y_i = (1 - \varepsilon_i)T(y_i) + \varepsilon_iU(y_i), \quad (8)$$

with $\varepsilon_i \in (0, 1]$, $\{\varepsilon_i\}$ monotonically decreasing to zero. If the two sequences $\{\varepsilon_m\}$ and $\{\alpha_m\}$, contained in $(0, 1]$, satisfy with some strictly increasing sequence $\{m(k)\}$ of positive integers

$$\liminf_{k \to \infty} \varepsilon_m(k) \geq \sum_{j = m(k)}^{m(k+1)} \alpha_j > 0, \quad (9)$$

$$\lim_{k \to \infty} [\varepsilon_m(k) - \varepsilon_m(k+1)] \sum_{j = m(k)}^{m(k+1)} \alpha_j = 0, \quad (10)$$

then the sequence $\{z_m\}$ generated by (7) converges to $p$.

**Proof.** We follow the pattern of proof in [4, pp. 117–119], but we dispense with inner product structure.
Banach’s fixed point theorem guarantees existence and uniqueness of each $y_i$. We calculate for $m > i > 1$

$$|z_m - y_i| = |\alpha_{m-1}(1 - \epsilon_{m-1})T(z_{m-1}) + \alpha_{m-1}\epsilon_{m-1}U(z_{m-1}) + (1 - \alpha_{m-1})z_{m-1} - y_i|$$

$$\leq (1 - \alpha_{m-1})|z_{m-1} - y_i| + \alpha_{m-1}|(1 - \epsilon_{m-1})T(z_{m-1}) + \epsilon_{m-1}U(z_{m-1}) - (1 - \epsilon_i)T(y_i) - \epsilon_i U(y_i)|$$

$$\leq [1 - \alpha_{m-1} + \alpha_{m-1}(1 - \epsilon_i + \alpha_{m-1}\epsilon_i q)] \cdot |z_{m-1} - y_i| + \alpha_{m-1} (\epsilon_i - \epsilon_{m-1}) \cdot |T(z_{m-1}) - U(z_{m-1})|.$$

Hence

$$|z_m - y_i| \leq [1 - \alpha_{m-1}\epsilon_i (1 - q)] \cdot |z_{m-1} - y_i| + \alpha_{m-1} (\epsilon_i - \epsilon_{m-1})c, \quad (11)$$

with some constant $c$, because $T$ and $U$ are self-mappings of the bounded set $C$. Since the exp function is convex and therefore $\exp(t) - 1 > t$ holds, it follows that

$$|z_m - y_i| \leq \exp[-\alpha_{m-1}\epsilon_i (1 - q)] \cdot |z_{m-1} - y_i| + \alpha_{m-1} (\epsilon_i - \epsilon_{m-1}).$$

By induction we conclude

$$|z_m - y_i| \leq \exp[-\epsilon_i (1 - q) \sum_{j=i}^{m-1} \alpha_j] \cdot |z_i - y_i| + c \sum_{j=i}^{m-1} \alpha_j (\epsilon_i - \epsilon_j).$$

On account of $\epsilon_j - \epsilon_i < \epsilon_i - \epsilon_m$ for $j < m$ we weaken this estimate to

$$|z_m - y_i| \leq \exp[-(1 - q)\epsilon_i \sum_{j=i}^{m-1} \alpha_j] \cdot |z_i - y_i| + c (\epsilon_i - \epsilon_m) \sum_{j=i}^{m} \alpha_j.$$

Starting with this inequality, which corresponds to inequality (12) in [4], one can easily adapt the arguments in [4, pp. 118–119] to conclude the proof. The details are omitted.

Examples of sequences $\{\alpha_n\}$ and $\{\epsilon_n\}$ that satisfy both the conditions (9) and (10) are given by $\alpha_n = 1/n$, and $\epsilon_n = 1/\log \log n$ for $n > 2$, or by $\alpha_n = n^{-p}$ and $\epsilon_n = n^{-q}$ for $n > 2$, provided $0 < p < 1$ and $0 < q < 1 - p$ holds (see Brück [4, p. 125]). Also one can choose $\alpha_n = \lambda \in (0, 1]$ fixed, and $\epsilon_n(k) = \epsilon_n(k+1) = \cdots = \epsilon_n(n+1-1) = k^{1 - p}$, where $n(k) \sim k^p$ and $p > 1$. The resulting iteration process is then related to [6, Theorem 4]. Furthermore the choice $\alpha_n = \lambda$, $\epsilon_n = n^{-p}(1 + n^{-p})^{-1}$, $p \in (0, 1)$ with $n(k) \sim k^r$, $r = (1 - p)^{-1}$ leads to the example D of Theorem 3D with $P_K = I$ in [2].

Simpler sufficiency criteria for strong convergence are provided by

**Theorem 4.** Let $C$ be a bounded closed convex subset of $E$. Suppose, the sequence $\{y_i\}$, given by (8), converges to a fixed point $p$ of $T$, where $\epsilon_i \in (0, 1]$ and $\lim_{i \to \infty} \epsilon_i = 0$. If the two sequences $\{\epsilon_i\}$ and $\{\alpha_i\}$, contained in $(0, 1]$, satisfy

$$\sum_i \alpha_i \epsilon_i = +\infty, \quad (12)$$
then the sequence \( \{z_m\} \) generated by (7) converges to \( p \).

**Proof.** We simplify (11) to

\[
\delta_{m+1} := |z_{m+1} - y_m| \leq \left[ 1 - \alpha_m \epsilon_m (1 - q) \right] |z_m - y_m|.
\]

On the other hand we have

\[
|y_i - y_{i-1}| \leq (1 - \epsilon_i) |y_i - y_{i-1}| + |\epsilon_{i-1} - \epsilon_i| \cdot |T(y_{i-1}) + \epsilon_i q y_i - y_{i-1}|
\]

\[+ |\epsilon_{i-1} - \epsilon_i| \cdot |U(y_{i-1})|,
\]

hence with some constant \( d \)

\[
|y_i - y_{i-1}| \leq d \cdot |\epsilon_{i-1} - \epsilon_i| \epsilon_i^{-1}.
\]

Thus we obtain with \( \chi_m = (1 - q) \alpha_m \epsilon_m, \gamma_m = d \cdot |\epsilon_m - \epsilon_m| \cdot \epsilon_m^{-1} \cdot \chi_m \)

\[
\delta_{m+1} \leq (1 - \chi_m) \delta_m + \chi_m \gamma_m,
\]

and consequently for arbitrary \( j \geq 0 \)

\[
\delta_{m+j+1} \leq \left( \prod_{i=m}^{m+j} (1 - \chi_i) \right) \delta_m + \sum_{i=m}^{m+j} \left( \prod_{k=i+1}^{m+j} (1 - \chi_k) \right) \chi_i y_i.
\]

By (12), \( \Pi(1 - \chi_j) \) diverges to zero. Since

\[
\sum_{i=m}^{m+j} \left( \prod_{k=i+1}^{m+j} (1 - \chi_k) \right) \chi_i \leq 1
\]

for any \( j \) and \( \lim_{i \to \infty} \gamma_i = 0 \) by (13), a well-known theorem of Toeplitz (cf. [8, p. 75]) implies that the second term in (14) converges to zero \( (j \to \infty) \), too. Thus we arrive at \( \lim_{i \to \infty} \delta_i = 0 \).

This result is closely related to Theorem 3D in [2] and contains (choose \( U(z) = y \) fixed, \( \alpha_j = 1 \) fixed) a recent result of Lions [9, Theorem 1].

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**References**


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