EULER CHARACTERISTICS AND CODIMENSIONS OF COMPLETE INTERSECTIONS

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ABSTRACT. Studies on relations between Euler characteristics and codimensions of complete intersections.

Let $F_1, F_2, \ldots, F_r$ be nonsingular hypersurfaces of degrees $a_1, a_2, \ldots, a_r$, in complex projective space $\mathbb{C}P^n$, and suppose that these hypersurfaces are in general position. The intersection $F_1 \cap F_2 \cap \cdots \cap F_r$ is a nonsingular algebraic variety denoted by $V_n[a_1, \ldots, a_r]$. In this short note, we prove the following theorem which completes the solution to the problem studied in [1]. The presentation of the proof follows closely that of the proofs in [1].

**Theorem.** Let $V_n$ be an n-dimensional complete intersection with Euler characteristic $\chi(V_n) = v_1 \cdots v_p$ for some prime numbers $v_1, \ldots, v_p$ ($\neq \pm 1$). Then $V_n$ can be imbedded in $\mathbb{C}P^{n+p-1}$ as a complete intersection except when $V_n$ is $V_1[2]$ or $V_1[2, 3]$ or $V_1[2, 2, 2]$.

**Proof.** In [2], Hirzebruch proved the following identity:

$$
\sum_{n=0}^{\infty} \chi(V_n[a_1, a_2, \ldots, a_r])z^n = \frac{a_1a_2 \cdots a_r}{(1 - z)^2} \prod_{i=1}^{r} \frac{1}{1 + (a_i - 1)z}.
$$

(1)

By multiplying power series, (1) implies

$$
\chi(V_n[a]) = \frac{(1 - a)^{n+2} - 1 + (n + 2)a}{a^2} \cdot a,
$$

(2)

$$
(-1)^n \chi(V_n[a_1, a_2, \ldots, a_r]) = a_r \sum_{k=0}^{n} (a_r - 1)^{n-k}(-1)^k \chi(V_k[a_1, a_2, \ldots, a_{r-1}]).
$$

(3)

By induction, we find

$$
(-1)^n \chi(V_n[a_1, a_2, \ldots, a_r]) = a_1a_2 \cdots a_r h_n[a_1, a_2, \ldots, a_r],
$$

(4)

where

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\[ h_n[a_1, a_2, \ldots, a_r] = \sum_{k_r=0}^{n} \sum_{k_{r-1}=0}^{k_r} \cdots \sum_{k_2=0}^{k_3} (-1)^{k_r}(a_r - 1)^{n-k_r} \]

\[ \cdots (a_2 - 1)^{k_3-k_2} \Delta_{k_2}[a_1] , \]

and \( \Delta_k[a] = \{(1-a)^{k+2} - 1 + (k+2)a\}/a^2 \). It is clear that \( h_n[a] = (-1)^n \Delta_n[a], \Delta_1[3] = 0, \Delta_0[a] = 1, \Delta_n[2] = (n+2)/2 \) when \( n \) is even, \( \Delta_n[2] = (n+1)/2 \) when \( n \) is odd, and \( (-1)^n \Delta_n[a] > 0 \) when \( n > 2 \) and \( a > 3 \). Thus, we obtain

\[ h_n[a_1, a_2, \ldots, a_r] > 1 \text{ for } r > 2, \text{ except } n = 1, r = 2 \]

\[ a_1 < 2, a_2 < 3, \text{ or } n = 1, r = 3, a_1, a_2, a_3 < 2. \]  \( \tag{5} \)

\[ h_n[a] \neq \pm 1 \text{ except } n = 1, a = 2 \text{ or } 4. \]  \( \tag{6} \)

Now, we assume that \( V_n[a_1, a_2, \ldots, a_r] \); \( a_1, a_2, \ldots, a_r > 2 \), is a complete intersection with \( \chi(V_n) = \nu_1 \cdots \nu_p \) for some prime integers \( \nu_1, \ldots, \nu_p \neq \pm 1 \).

If \( r < p \), then it is done. If \( r > p \), then \( (4) \) implies \( r = p \) and \( h_n[a_1, a_2, \ldots, a_r] = \pm 1 \). From (5) and (6) we see that this is impossible unless \( n = 1 \) and \( V_n \) is one of the following: \( V_{1}[2, 2], V_{1}[2, 3], V_{1}[2, 2, 2], V_{1}[2] \) or \( V_{1}[4] \). Since we have \( \chi(V_{1}[2, 2]) = 0, \chi(V_{1}[2, 3]) = -6, \chi(V_{1}[2, 2, 2]) = -8, \chi(V_{1}[2]) = 2 \) and \( \chi(V_{1}[4]) = -4 \), the theorem is proved.

**References**


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