ON GROUPOIDS DEFINED BY COMMUTATORS

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Abstract. We study matrices $R, L$ which count the numbers of solutions of $ix = j$ and $xi = j$. For slight generalizations of $R, L$, the relation $RL = LR$ characterizes associativity of a groupoid. For groupoids defined by group commutators $xyx^{-1}y^{-1}$ the relation $RL = LR$ is valid. In addition one can study analogues of Green's relations. Any $\mathcal{H}$-class contains at most four $\mathcal{H}$-classes in a commutator groupoid.

In this paper we mainly consider groupoids whose underlying set is a group, with groupoid multiplication $x \ast y = xyx^{-1}y^{-1}$. Our interest is mainly in the matrices $R$ and $L$ such that $r_{ij}$ counts the number of solutions of $i \ast x = j$ and $l_{ij}$ counts the number of solutions of $x \ast i = j$.

Definition. Let $G$ be a groupoid. Let $t, u$ be functions from $G$ to a commutative semiring $K$ with 0. Then $R(t)$ is the matrix $(r_{ij})$ for $i, j \in G$ such that $r_{ij} = \sum t(x)$, the summation being over all $x$ such that $ix = j$, if this sum is defined. And $L(u)$ is the matrix $(l_{ij})$ such that $l_{ij} = \sum u(x)$, the summation being over all $x$ such that $xi = j$ if this sum is defined. Summations over the empty set are considered to be 0. And we assume $0 + k = k$ and $0k = 0$ for all $k \in K$.

In this paper we consider the two cases: (i) $G$ finite, $K = \mathbb{Z}^+ \cup \{0\}$; (ii) $G$ arbitrary, $K$ the Boolean algebra $\{0, 1\}$. The following proposition is essentially due to M. S. Putcha [2].

Proposition 1. In the two cases just mentioned, the matrices $R(t), L(u)$ commute for all $t, u$ if and only if $G$ is associative.

Proof. We have

$$(R(t)L(u))_{ij} = \sum t(x)u(y)$$

where the summation is over all pairs such that $ix = k, yk = j$ for some $k$, i.e. all pairs such that $y(ix) = j$. Likewise

$$(L(u)R(t))_{ij} = \sum u(y)t(x)$$

where the summation is over all pairs such that $(yi)x = j$. So if $G$ is...
associative $R(t)$, $L(u)$ commute. For the converse, let $u$, $t$ range independently over all functions which send every element of $G$ except one, to zero. This proves the proposition.

**Remark.** By choosing $t$, $u$ to send elements of $G$ to randomly chosen real numbers, this might give a quick computer test for nonassociativity of a groupoid.

From here on, we assume both $t$, $u$ send all elements of $G$ to 1, and we write $R$, $L$ for $R(t)$, $L(u)$.

**Definition.** A group commutator groupoid is a groupoid $G$ whose underlying set is a group and whose groupoid product is given by $xyx^{-1}y^{-1}$.

**Proposition 2.** Let $G$ be a group commutator groupoid. Let $T$ be the matrix of the permutation $x \rightarrow x^{-1}$. Then $RT = TR = L$. Therefore $R$, $L$ commute.

**Proof.** The equation $RT = L$ follows from $(ix^{-1}-1)^{-1} = xix^{-1}i^{-1}$. The identity $i^{-1}ix^{-1} = (i^{-1}xi)(i^{-1}xi)^{-1}i^{-1}$ implies $TR = RT$.

**Proposition 3.** For each $a$, $b$, $R_{ab}$ and $L_{ab}$ are each either zero or the order of the centralizer of $a$. The row sums of $R$, $L$ all equal the order of $G$. The $b$th column sum of $R$ and the $b$th column sum of $L$ each equal the number of pairs $x$, $y$ such that $x^ay^{-1}y^{-1} = b$. The trace of $R$ equals the sum of the orders of the centralizers of those elements $a$ which are conjugate to $a^2$.

**Proof.** The entry $R_{ab}$ is the number of solutions of $xa^{-1}x^{-1} = a^{-1}b$. This is either zero or has the same order as the centralizer of $a^{-1}$. But the centralizer of $a$ equals the centralizer of $a^{-1}$. Likewise for $L_{ab}$. The second and third statements can be observed to be true. For the fourth statement, note that the trace of $R$ is the sum of the orders of the centralizers of such that $xa^{-1}x^{-1} = e$. But this can happen only if $a = e$. Likewise for $L$. This proves the proposition.

**Definition.** A (left, right) ideal in a groupoid is a subset closed under (left, right) multiplication. The principal (left, right) ideal generated by an element is the intersection of all (left, right) ideals containing that element. Two elements are ($\mathcal{R}$, $\mathcal{L}$, $\mathcal{F}$)-equivalent if and only if they generate the same principal (right, left, two-sided) ideal. They are $\mathcal{H}$-equivalent if and only if they are both $\mathcal{R}$- and $\mathcal{L}$-equivalent. These equivalence relations are called Green’s relations.

**Definition.** A directed graph is strongly connected if and only if every point can be reached from every other point by a directed path.

Corresponding to this one can express any graph as a disjoint union of its strong components. We consider the graph of a matrix to be the graph whose vertices are the elements of the index set of the matrix, having an edge from $i$ to $j$ if and only if the $(i, j)$-entry of the matrix is nonzero.

**Proposition 4.** For any groupoid, the strong components of the graphs of
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I + R, I + L, (I + R)(I + L) are the $R$, $L$, $\mathcal{J}$-classes. Here $I$ denotes the identity matrix.

Note that if the elements of $G$ are arranged in the order of an ascending chain of normal subgroups, the matrices $R, L$ will assume a block triangular form. In addition nilpotency can easily be detected.

**Theorem 5.** A finite group $G$ is nilpotent if and only if the matrix $R$ of its commutator groupoid is nilpotent. Likewise for $L$.

**Proof.** Suppose $G$ is nilpotent. Arrange the elements of $G$ in the order of an ascending central series. Then $R, L$ are lower subtriangular matrices.

Suppose $G$ is not nilpotent. Then by Theorem 14.4.7 of [1] there exist $x, p$ such that $x$ has order prime to $p$ and $x$ normalizes but does not centralize some $p$ subgroup $Q$. Choose $Q$ to be minimal. Then $x$ acts trivially on $[Q, Q]$ by conjugation. Then $x$ does not act trivially on $Q/[Q, Q]$ by conjugation, or the group generated by $x, Q$ would have a central series. So $x$ gives a nontrivial automorphism of $Q/[Q, Q]$. An endomorphism of $Q/[Q, Q]$ is given by $y \to xyx^{-1}y^{-1}$, mod$[Q, Q]$. If this endomorphism were nilpotent, the automorphism $xyx^{-1}$ would have order a power of $p$, which is false. Thus the endomorphism of $Q/[Q, Q]$ given by $y \to xyx^{-1}y^{-1}$ is not nilpotent. This implies $L$ is nonnilpotent. Similarly for $R$.

**Theorem 6.** If $G$ is a group commutator groupoid, every $\mathcal{J}$-class of $G$ contains at most two $\mathcal{R}$-classes and at most two $\mathcal{L}$-classes. If there are two of either type, they are equal in size. And $a \mathcal{J} b$ if and only if there exists $c$ such that $a \mathcal{R} c, c \mathcal{L} b$ if and only if there exists $d$ such that $a \mathcal{R} d, d \mathcal{R} b$.

**Proof.** The classes will not be affected if we use matrices over the Boolean algebra $\{0, 1\}$ always, The classes obtained from $I + R, I + L, (I + R)(I + L)$ are the same as those obtained from

\[
\begin{align*}
\overline{R} &= I + R + R^2 + \ldots, \\
\overline{L} &= I + L + L^2 + \ldots, \\
\overline{RL} &= \sum_0^\infty R^n + \sum_1^\infty R^nT.
\end{align*}
\]

Suppose $a \not\mathcal{J} b$. Note that $\overline{R}, \overline{L}, \overline{RL}$ are idempotent. Thus there is an edge in the graph of $\overline{RL}$ from $a$ to $b$ and one from $b$ to $a$. Each of these two edges comes from one of the two summands

\[
\sum_0^\infty R^n, \quad \sum_1^\infty R^nT.
\]

In the first case there is an $\overline{R}$ edge from one to the other and in the second case there is an $\overline{R}$ edge from one to the inverse of the other. We denote the existence of an edge from one to the other by $\to$. We observe that $x \to y^{-1}$ if and only if $x^{-1} \to y$ since $RT = TR$. There are four cases:

Case 1. $a \to b, b \to a$ in the graph of $\overline{R}$. Then $a \mathcal{R} b$. 

Case 2. \( a \to b^{-1} \), \( b \to a^{-1} \) in the graph of \( \overline{R} \). Then \( a \not\equiv b \).

Case 3. \( a \to b \), \( b \to a^{-1} \) in the graph of \( \overline{R} \). Then also \( a^{-1} \to b^{-1} \), \( b^{-1} \to a \).

These imply \( a \not\equiv b \).

Case 4. \( a \to b^{-1} \), \( b \to a \). Again \( a \not\equiv b \). Therefore either \( a \) lies in the \( \equiv \)-class of \( b \) or that of \( b^{-1} \). Thus the \( \equiv \)-class of \( b \) contains at most two \( \equiv \)-classes. Likewise it contains at most two \( \mathcal{L} \)-classes.

Suppose there do exist two \( \equiv \)-classes in some \( \mathcal{L} \)-class. Then there exist \( a, b \) such that \( a \not\equiv b \) but not \( a \equiv b \). Thus the situation must be that of Case 2. And for any \( a, b \) in different \( \equiv \)-classes but in the same \( \mathcal{L} \)-class, this must be so. Therefore \( a \not\equiv b^{-1} \). Thus for any \( b \) in this \( \mathcal{L} \)-class, \( b \) and \( b^{-1} \) will lie in different \( \equiv \)-classes. Therefore the mapping \( x \to x^{-1} \) will be a 1-1 onto mapping from one \( \equiv \)-class to the other. Likewise for \( \mathcal{L} \)-classes.

In Cases 1, 3, 4, \( a \equiv b \) and the last statement is valid. Suppose we are in the second case. Suppose \( a \to b^{-1} \) by an odd number of edges in the graph of \( R \), and \( b^{-1} \to a \) by an odd number. Then since \( L = RT \), \( a \not\equiv b \). Suppose \( a \to b^{-1} \) by an even number of \( \equiv \) edges and \( b^{-1} \to a \) by an even number. Let \( a \to x \) be the first edge in the sequence from \( a \) to \( b^{-1} \). Then \( a \to x \to b^{-1} \to a \to x \). So \( a \equiv x, x \not\equiv b \). And \( a \not\equiv x^{-1}, x^{-1} \equiv b \). If the number of edges from \( a \) to \( b^{-1} \) is even and the number of edges from \( b^{-1} \) to \( a \) is odd, or vice versa, we can double the path and obtain one of the two former cases. This proves the theorem.

Example 1. For the symmetric group on three symbols, \( L \) and \( R \) are, respectively

\[
\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 \\
3 & 0 & 3 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0
\end{bmatrix}
\quad
\begin{bmatrix}
6 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 3 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & 2 & 0 & 0 & 0
\end{bmatrix}
\]

Example 2. It is difficult to find a finite group with a \( \mathcal{J} \)-class containing four different \( \mathcal{K} \)-classes. Consider the semidirect product of the multiplicative group of numbers of the form \( \pi^i (\pi - 1)^j \) with the additive real numbers. Then \( 1 \not\equiv \pi - 1 \) but \( 1 \) and \( \pi - 1 \) are not \( \equiv \)-equivalent. Also \( 1 \equiv 1 - \pi \) but \( 1 \) and \( 1 - \pi \) are not \( \equiv \)-equivalent. Then Theorem 6 implies there are at least four distinct \( \mathcal{K} \)-classes, in the \( \mathcal{J} \)-class of 1.

Remark. Many of the results demonstrated here are trivially true for groupoids defined by Lie algebra commutators.

References


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