ASCENT, DESCENT AND COMPACT PERTURBATIONS

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Abstract. The collections of upper semi-Fredholm operators with finite ascent and of lower semi-Fredholm operators with finite descent are both closed under commuting compact perturbations.

Suppose that $T$ and $V$ are commuting bounded linear operators on the Banach space $X$ and that $T - V$ is compact. In Theorem 2 below we show that if $T$ is upper semi-Fredholm, then $T$ has finite ascent if and only if $V$ does; and, dually, if $T$ is lower semi-Fredholm, it has finite descent if and only if $V$ has finite descent. If $T$ is invertible or just Fredholm of index 0, it has long been known that $V$ has finite ascent and descent [3, Theorem 6.3, p. 610], [1, Theorem (1.4.5), p. 12], [2, pp. 39–42]; but, even in this special case, examples show that the commutativity of $T$ and $V$ is crucial [3, p. 599], [1, pp. 13–14], [2, p. 40].

We start with a lemma which treats the special case that $T$ is onto.

Lemma 1. Suppose that $T$ and $V$ are commuting bounded linear operators on the Banach space $X$. If $T - V$ is compact and $T$ is onto, then $V$ has finite descent.

Proof. For each nonnegative integer $k$, the range, $R(V^k)$, has finite codimension [1, Corollary (1.3.7)(b), p. 9] and the map induced by $T$ on $X/R(V^k)$ is onto. Therefore this induced map is one-to-one, so that the null-space $N(T) \subseteq R(V^k)$. Since $T$ is onto, there is a positive number $\gamma$ for which $\|Tx\| > \gamma \text{ dist}(x, N(T))$ for all $x$ in $X$. Suppose that $x$ belongs to $X$ and $z$ belongs to $R(V^k)$; then $T(R(V^k)) = R(V^kT) = R(V^k)$ so there is a $y$ in $R(V^k)$ with $Ty = z$. Thus we have $\|Tx - z\| = \|T(x - y)\| > \gamma \text{ dist}(x - y, N(T)) > \gamma \text{ dist}(x, R(V^k))$, since $N(T) \subseteq R(V^k)$. Since this holds for all $z$ in $R(V^k)$, we obtain

$$\text{dist}(Tx, R(V^k)) > \gamma \text{ dist}(x, R(V^k)).$$

Suppose $V$ had infinite descent. Then there would be a bounded sequence $\{x_n\}$ with $x_n \in R(V^n)$ and $\text{dist}(x_n, R(V^{n+1})) > 1$. Let $K = T - V$ and suppose $m > n$. Then $Kx_m - Kx_n = (Kx_m + (T - K)x_n) - Tx_n$. So that

$$\|Kx_m - Kx_n\| > \text{dist}(Tx_n, R(V^{n+1})) > \gamma \text{ dist}(x_n, R(V^{n+1})) > \gamma.$$
But this contradicts the compactness of $K$, so $V$ must have finite descent.

**Theorem 2.** Suppose that $T$ and $V$ are commuting bounded linear operators on the Banach space $X$ with $T - V$ compact.

(A) If $T$ is upper semi-Fredholm, then $V$ has finite ascent if and only if $T$ has finite ascent.

(B) If $T$ is lower semi-Fredholm, then $V$ has finite descent if and only if $T$ has finite descent.

**Proof.** Suppose first that $T$ is lower semi-Fredholm. Since $V$ is also lower semi-Fredholm [1, Corollary (1.3.7)(b), p. 9], it will be enough to show that $V$ has finite descent if $T$ has. Let $p$ be an integer with $R(T^p) = R(T^{p+1})$. Then $R(T^p)$ is a closed subspace of finite codimension [1, Corollary (1.3.3), p. 9] and the restriction of $T$ to $R(T^p)$ is onto. Therefore, by Lemma 1, the restriction of $V$ to $R(T^p)$ has finite descent, so that there is an integer $k$ for which

$$R(V^m) \subseteq R(V^mT^p) = R(V^kT^p)$$

for all $m > k$. Since $R(V^kT^p)$ has finite codimension [1, Corollary (1.3.3), p. 9], $V$ has finite descent. This proves (B).

Now suppose that $T$ and $V$ are upper semi-Fredholm. Then $T^*$ and $V^*$ are lower semi-Fredholm, and the ascent of $T$ and $V$, respectively, equals the descent of $T^*$ and of $V^*$, respectively [1, pp. 7–8]. Part (A) now follows directly from Part (B).

Instead of proving Part (A) from Part (B) in Theorem 2 by duality arguments, we could have proved the dual result to Lemma 1 for $T$ bounded below and then used this result to prove Part (A). The direct proofs are very similar to our proofs of Lemma 1 and Theorem 2(B).

In subsequent papers we will use the results of the present paper to study compact perturbations of more general classes of operators.

**References**


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