A COUNTEREXAMPLE TO A CONJECTURE OF A. H. STONE

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Abstract. A. H. Stone has offered a sequence, \( S(n); n > 2 \), of conjectures characterizing multicoherence for locally connected, connected, normal spaces. The conjecture \( S(n) \) is, "\( X \) is multicoherent if and only if \( X \) can be represented as the union of a circular chain of continua containing exactly \( n \) elements". It is known that \( S(3) \) always obtains and that \( S(6) \) obtains if the space is compact. In this paper, we construct a multicoherent plane Peano continuum \( C \) for which \( S(7) \) fails. Since \( S(n + 1) \) implies \( S(n) \), \( n > 2 \), \( S(n) \) fails for \( C \) for all \( n > 6 \). Furthermore we show that for any integer \( n > 3 \) there exists a plane Peano continuum for which \( S(2n) \) obtains while \( S(2n + 1) \) fails.

Introduction. Throughout this paper \( X \) will denote a locally connected, connected normal space. By a continuum we mean a closed and connected (not necessarily compact) subset of \( X \). For \( A \subset X \), \( b_0(A) \) denotes the number of components of \( A \) less one (or \( \infty \) if this number is infinite). The degree of multicoherence, \( r(X) \), of \( X \) is defined by

\[
r(X) = \sup\{ b_0(H \cap K): X = H \cup K \text{ and } H \text{ and } K \text{ are subcontinua of } X \}.
\]

If \( r(X) = 0 \), \( X \) is said to be unicoherent and we say that \( X \) is multicoherent otherwise. By a chain \( \kappa \) in \( X \) we mean a finite collection of subcontinua of \( X \) that can be ordered \( \kappa = \{ K_1, K_2, \ldots, K_n \} \) so that \( K_i \cap K_j \neq \emptyset \) if and only if \( |i - j| < 1 \). A circular chain in \( X \) is a collection of subcontinua \( \kappa \) such that no three members of \( \kappa \) have a point in common and if \( K \in \kappa \), then \( \kappa - \{ K \} \) is a chain in \( X \). Let \( n > 2 \) be an integer and let \( S(n) \) denote the following statement:

\( S(n): X \) is multicoherent if and only if \( X \) can be represented as the union of a circular chain containing exactly \( n \) elements.

In a private communication, A. H. Stone conjectured that \( S(n) \) is true for all \( n > 2 \) and he stated that he had established \( S(n) \) for all \( n > 2 \) whenever \( 0 < r(X) < \infty \). A. D. Wallace established \( S(3) \) for Peano continua in [4]. A. H. Stone announced \( S(3) \) for locally connected normal spaces in [3] and in [2], the second author included a proof of \( S(3) \) for such spaces. In [1], the second author showed that \( S(4) \) obtains for a large class of spaces and in [2], he showed that \( S(6) \) always obtains if \( X \) is compact. The purpose of this note
is to give an example of a multicoherent plane Peano continuum for which $S(7)$ fails.

**Lemma 9.** Let $a$ and $b$ be distinct points in $X$ and suppose that $X - \{a, b\} = R \cup P \cup Q$ where $R$, $P$ and $Q$ are pairwise disjoint open connected sets and $\bar{R} \cap \bar{P} \cap \bar{Q} = \{a, b\}$. If $\kappa$ is a circular chain in $X$ and $\cup \kappa = X$ and some $K \in \kappa$ lies entirely in $R$, then $(\bar{P} \cup \bar{Q})$ meets at most four elements of $\kappa$.

**Proof.** Notice that $\{a, b\}$ is the boundary of each of $P$, $Q$ and $R$. Let $K_0 \subset R$ and $\{K_1, K_2, \ldots, K_n\}$ be a chain representation of $\kappa - \{K_0\}$. Let $S$ be the union of those $K \in \kappa$ that contain $a$, let $T$ be the union of those $K \in \kappa$ that contain $b$, and let $V$ be the union of those $K \in \kappa$ that contain neither $a$ or $b$. Clearly, $V$ has at most two components, one of which contains $K_0$ and is therefore contained in $R$. It follows that either $\bar{P}$ or $\bar{Q}$ must fail to intersect $V$. Therefore either $\bar{Q} \subset S \cup T$ or $\bar{P} \subset S \cup T$. Since both $P$ and $Q$ are connected and intersect both $S$ and $T$ it follows that $S \cap T \neq \emptyset$. That is, there is a $k > 0$ such that $\{a, b\} \subset K_k \cup K_{k+1}$. It follows that $K_i \subset R$ except possibly when $i = k - 1, k, k + 1$ or $k + 2$.

**Construction of the example.** For each positive integer $i$, let $C_i$ be a one-dimensional simplicial complex in the plane, with $V_i$ as its set of vertices and $\mathcal{E}_i$ as its set of edges, constructed as follows:

Let $V_1$ consist of three evenly distributed points on the unit circle $\{z: |z| = 1\}$ and let $\mathcal{E}_1$ consist of three connecting open intervals. Then $C_1 = \cup \{I: I \in \mathcal{E}_1\} \cup V_1$. Let $U_1 = \{U(I): I \in \mathcal{E}_1\}$ be a set of mutually disjoint bounded open convex sets such that $I \subset U(I)$ for $I \in \mathcal{E}_1$. Suppose $C_{n}, V_n, U_n$, and $\mathcal{E}_n$ have been defined. For each $I \in \mathcal{E}_n$ let $a(I), b(I)$ be the endpoints of $I$, and $m(I)$ its midpoint $(a(I) + b(I))/2$. Let $t_n$ be chosen so that $1 < t_n < 1 + 1/n$ and (for all $I \in \mathcal{E}_n$) the two “half-open” line segments $(a(I), t_nm(I)), [t_nm(I), b(I))$ are both contained in $U(I)$. Let $V_{n+1} = V_n \cup \{m(I): I \in \mathcal{E}_n\} \cup \{t_nm(I): I \in \mathcal{E}_n\}$. Let $\mathcal{E}_{n+1} = \{(a(I), m(I)): I \in \mathcal{E}_n\} \cup \{(m(I), b(I)): I \in \mathcal{E}_n\} \cup \{(a(I), t_nm(I)): I \in \mathcal{E}_n\} \cup \{(t_nm(I), b(I)): I \in \mathcal{E}_n\}$. Then $C_{n+1} = \cup \{I: I \in \mathcal{E}_{n+1}\} \cup V_{n+1}$. Let $\mathcal{U}_{n+1} = \{U(I): I \in \mathcal{E}_{n+1}\}$ be a set of mutually disjoint open convex sets such that if $I \in \mathcal{U}_{n+1}$ and $J \in \mathcal{U}_n$ and $I \subset U(J)$ then $I \subset U(I) \subset U(J)$. Let $D = \cup_{i=1}^{\infty} C_i$ and let $C = D$.

It is clear that $C$ is multicoherent. By Theorems 3 and 6 of [2], $S(6)$ obtains for $C$. We will now show that $S(7)$ fails for $C$. (Since $(S_n + 1)$ always implies $S(n), S(k)$ for $k > 6$ fails for $C$.)

**Definition.** For $I = (a(I), b(I)) \in \mathcal{E}_n$ let $I' = (a(I), t_n m(I)) \cup [t_n m(I), b(I))$; we use the convention $(I')' = I$. Let $\mathcal{E}_n' = \{I': I \in \mathcal{E}_n\} \cup \mathcal{E}_n$.

The following lemma seems clear.

**Lemma A.** (1) $C$ is a Peano continuum.

(2) If $I \in \mathcal{E}_n'$ has endpoints $a(I)$ and $b(I)$ then $C - \{a(I), b(I)\}$ has exactly three components $P(I), P(I')$ and $Q(I)$ where $I \subset P(I)$ and $I' \subset P(I')$.

(3) If, for each $i$, $I_i \in \mathcal{E}_i$ then $\lim_{i \to \infty} \text{dia}(I_i) = \lim_{i \to \infty} \text{dia} P(I_i) = 0$. 

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Theorem 1. The plane Peano continuum $C$ cannot be the union of a circular chain with seven elements.

Proof. Suppose to the contrary that $\kappa$ is a circular chain with seven elements and $\bigcup \kappa = C$. Since each of the three vertices of $C_1$ is contained in at most two $K \in \kappa$ it follows that some $K \in \kappa$ contains no vertex of $C_1$ and is therefore contained in $P(I_1)$ for some $I_1 \in \mathcal{S}'$. If $a$ and $b$ are the endpoints of $I_1$, we may write $C - \{a, b\}$ as the union of $R = P(I_1)$, $P = P(I_1')$ and $Q = Q(I_1)$. According to Lemma 9, $(P \cup Q)$ meet at most four elements of $\kappa$ and so $R$ contains three elements, say $K_1^1$, $K_2^1$, and $K_3^1$. Now the vertex of $C_2$ that lies in $R$, must miss one of these $K_i^1$'s, say $K_1^1$. Then there is an $I_2 \in \mathcal{S}_2'$ such that $P(I_2)$ contains $K_1^1$. Let $a_2, b_2$ be the endpoints of $I_2$ and note that

$$C - \{a_2, b_2\} = P(I_2) \cup P(I_2') \cup Q(I_2).$$

Again by Lemma 8, $P(I_2)$ must contain 3 elements of $\kappa$ and hence the new vertex of $C_3$ that lies in $P(I_2)$ must miss one of these elements of $\kappa$. We continue as above and select a sequence $I_2, I_3, \ldots$, such that $I_j \in \mathcal{S}_j'$, $P(I_{j+1}) \subset P(I_j)$ and each $P(I_j)$ contains a member of $\kappa$, $K_j$. Since $\kappa$ is finite, there is a subsequence $\{I_{j_r}\}$ such that $K_{j_r} = K_{j_s}$ for all $r, s > 1$. But then $K_j \subset \cap_{n=2}^{\infty} P(I_n)$, and, by Lemma A, $K_j$ must be a singleton. Of course, this is impossible and this completes the proof.

Theorem 2. For all integers $n \geq 3$, there exists a plane Peano continuum $P(n)$ such that $S(2n)$ obtains but $S(2n + 1)$ fails.

Proof. We construct $P(n)$ in the same fashion we constructed $C$ except in this instance we change the example by letting $C_1$ be a regular $n$-gon. If $I$ is an edge in $C_1$ with endpoints $a$ and $b$ then $\overline{P(I)} \cup \overline{P(I')}$ can be easily written

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as the union of two continua $K_i^a$ and $K_i^b$ where $a \notin K_i^b$ and $b \notin K_i^a$. Then
\{ $K_i^a: I$ is an edge in $C_1$ \} \cup \{ $K_i^b: I$ is an edge in $C_1$ \} is a circular chain that
covers $P(n)$ and has $2n$ elements. Let $\kappa$ be any circular chain that covers
$P(n)$. If every element of $\kappa$ contains a vertex of $C_1$ then $\kappa$ has at most $2n$
elements. If some element of $\kappa$ fails to contain a vertex of $C_1$ then there is an
$I \in \mathcal{E}_i$ such that $P(I)$ contains an element of $\kappa$. The same proof used for
$P(3)$ then shows that in this case $\kappa$ has at most 6 elements.

Remark. The authors have not been able to construct an example of a
plane Peano continuum $P$ for which $S(2n + 1)$ obtains while $S(2n + 2)$ fails.

References

1. R. F. Dickman, Jr., Some mapping characterizations of unicoherence, Fund. Math. 78 (1973),
27-35.
427-447.

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