THE LEBESGUE DECOMPOSITION THEOREM FOR
PARTIALLY ORDERED SEMIGROUP-VALUED MEASURES

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Abstract. The present paper is concerned with partially ordered
semigroup-valued measures. Below are given generalizations of the classical
Lebesgue Decomposition Theorem.

These results can be applied to Stone or $W^*$ algebra-valued positive
measures (cf. [3], [12], [13], [14]).

1. Preliminaries. By a partially ordered semigroup $X$ we mean a commuta-
tive semigroup with identity 0, equipped by a partial ordering $\leq$, compatible
with the structure of $X$ under the conditions:

(i) If $x, y, z$ are elements of $X$ with $x \leq y$ ($x \leq y$ and $x \neq y$) then
$x + z \leq y + z$.

(ii) $x + \sup E = \sup (x + E)$, whenever there exist $\sup E$ (the supremum
of $E$ in $X$) and $\sup (x + E), E \subseteq X, x \in X$.

Now $X$ is monotone complete if every majorised increasing directed family
in $X$ has a supremum in $X$. Moreover, $X$ is of the countable type if every
subset $E$ of $X$ that has a supremum in $X$, contains a countable subset $E^* \subseteq E$
so that: $\sup E = \sup E^*$.

Let $X$ be a partially ordered semigroup and $H$ a ring of subsets of $T$. The
function $m: H \rightarrow X$ is an $o$-measure (order measure) on $H$, if $m$ is positive on
$H (m(A) > 0, \text{ for every } A \in H)$ and $m(\bigcup_{n \in N} A_n) = \sup \{\sum_{i=1}^{n} m(A_n) : n \in N\}$ whenever $(A_n)_{n \in N}$ is a disjoint sequence of elements of $H$ with $(\bigcup_{n \in N} A_n) \in H$.

The following propositions can be easily proved.

Proposition 1.1. Let $m: H \rightarrow X$ be an $o$-measure on $H$.

(1) $m(\emptyset) = 0$.

(2) $m$ is finitely additive on $H$ and $m(A) \leq m(B)$, whenever $A, B \in H$ with
$A \subseteq B$.

(3) For every sequence $(A_n)_{n \in N}$ in $H$ with $(\bigcup_{n \in N} A_n) \in H$ and
$\sup \{\sum_{i=1}^{n} m(A_i) : n \in N\} \in X$, implies: $m(\bigcup_{n \in N} A_n) \leq \sup \{\sum_{i=1}^{n} m(A_i) : n \in N\}$. 

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(4) If $X$ is monotone complete then for every disjoint family $(A_i)_{i \in I}$ in $H$ with $\bigcup_{i \in I} A_i \in H$ implies: $m(\bigcup_{i \in I} A_i) \geq \sum_{i \in I} m(A_i) = \sup \{ \sum_{i \in J} m(A_i): J \subseteq I, J \text{ finite} \}$.  

**Proposition 1.2.** The function $m: H \to X$ is an $\omega$-measure on $H$ if and only if $m$ is positive, finitely additive on $H$ and $m(A_n) \uparrow m(A)$ $(m(A_n) < m(A_{n+1})$, $n \in N$ and $m(A) = \sup \{ m(A_n): n \in N \}$), for every increasing sequence $(A_n)_{n \in N}$ in $H$ with $A_n \uparrow A \in H$.

2. Absolutely continuous and singular $\omega$-measures. Let $X$, $Y$ be partially ordered semigroups and let $m: H \to X$, $l: H \to Y$ be $\omega$-measures on $H$. $l$ is $m$-absolutely continuous on $H$ ($l \ll m$) if $l(A) = 0$ whenever $A \in H$ with $m(A) = 0$. On the other hand $l$ is $m$-singular on $H$, ($l \perp m$) if for every $A$ in $H$ there is $B$ in $H$: $B \subseteq A$, $m(B) = 0$ and $l(A - B) = 0$. So $m \perp l$ if and only if $l \perp m$.

The following proposition can be easily verified.

**Proposition 2.1.** Let $m, l: H \to X$, $k: H \to Y$ be $\omega$-measures on $H$.

1. If $l \ll m$ and $l \ll k$ then $l = 0$.
2. If $l \ll m$ and $k \ll m$ then $l \perp k$.
3. $l \perp l$ if and only if $l = 0$.
4. If $m \perp l$ and $m \perp k$ then $m \perp (l + k)$.
5. If $l \perp m$ and $k \perp m$ then $(l + k) \perp m$.
6. If $X = Y$, and $l \ll m + k$, $l \perp m$ then $l \ll k$.

On the other hand the following lemma will be useful in the sequence.

**Lemma 2.2.** Let $m_i: H \to X$, $i \in I$, be an increasing directed family of $\omega$-measures on $H$. Suppose, that $X$ is a monotone complete partially ordered semigroup and for every $A \in H$ there is $x$ in $X$ such that: $m_i(A) \leq x$, whenever $i \in I$. Then the function $m: H \to X$, $m(A) = \sup \{ m_i(A): i \in I \}$ is an $\omega$-measure on $H$.

**Proof.** Let $A, B \in H$ with $A \cap B = \emptyset$, so $m(A \cup B) = \sup \{ m_i(A \cup B): i \in I \} = \sup \{ m_i(A) + m_i(B): i \in I \} \leq \sup \{ m_i(A): i \in I \} + \sup \{ m_i(B): i \in I \} = m(A) + m(B)$. Furthermore let $i, j$ be any pair of indices. Then there exist $h \in I$ such that, $h > i$ and $h > j$, hence $m_i(A) + m_j(B) < m_h(A) + m_h(B) = m_h(A \cup B) < m(A \cup B)$, which implies $m(A) + m(B) = m(A \cup B)$, namely $m$ is finitely additive on $H$. Evidently $m(A) < m(B)$ whenever $A, B \in H$ with $A \subseteq B$.

Finally let $(A_n)_{n \in N}$ be a sequence in $H$ with $A_n \uparrow A \in H$. Then $m_i(A_n) \uparrow m_i(A)$, for every $i \in I$. Thus:

$$\sup \{ m(A_n): n \in N \} = \sup \{ \sup \{ m_i(A_n): i \in I \}: n \in N \}, \quad (1)$$

$$m(A) = \sup \{ \sup \{ m_i(A_n): n \in N \}: i \in I \}. \quad (2)$$

But $(m_i(A_n): i \in I, n \in N) = \bigcup_{i \in I} \{ m_i(A_n): n \in N \} = \bigcup_{n \in N} \{ m_i(A_n): i \in I \}$, hence.
\[ \sup \{ \sup \{ m_i(A_n) : i \in I \} : n \in N \} = \sup \{ \sup \{ m_i(A_n) : n \in N \} : i \in I \} \]
\[ = \sup \{ m_k(A_n) : i \in I, n \in N \} \] (3)

(cf. [11, p. 12, Theorem 1.6.1]). Therefore by (1), (2) and (3) it follows that \( m(A_n) \uparrow m(A) \) and the assertion follows from Proposition 1.2.

Hereafter by \( S \) it is denoted a \( \sigma \)-ring of subsets of \( T \).

**Proposition 2.3.** Let \( m_i : S \to X, i \in I \) be an increasing directed family of \( \sigma \)-measures on \( S \) and \( l : S \to Y \) be another \( \sigma \)-measure on \( S \). Suppose that \( X \) is of the countable type partially ordered semigroup, \( \sup \{ m_i(A) : i \in I \} = m(A) \in X \), whenever \( A \in S \) and \( m_i \perp l \) for every \( i \in I \). Then \( m : S \to X \) is an \( \sigma \)-measure on \( S \) with \( m \perp l \).

**Proof.** By Lemma 2.2 it follows that \( m \) is an \( \sigma \)-measure on \( S \). Now let \( A \in S \). Then there is a countable subset \( \{ i(n) : n \in N \} \) of \( I \), such that: \( m(A) = \sup \{ m_{i(n)}(A) : n \in N \} \). On the other hand, there is a sequence \( (B_n)_{n \in N} \) in \( S \) with \( B_n \subseteq A \) and \( m_{i(n)}(A) = m_{i(n)}(B_n) \) and \( l(B_n) = 0 \), for every \( n \in N \). We put \( B = \bigcup_{n \in N} B_n \) hence \( B \subseteq A \), \( m_{i(n)}(A) = m_{i(n)}(B) \) and \( l(B) = 0 \), \( n \in N \). Consequently

\[ m(A) = \sup \{ m_{i(n)}(A) : n \in N \} = \sup \{ m_{i(n)}(B) : n \in N \} < m(B) < m(A), \]

so \( m(A - B) = 0 \) and \( l(B) = 0 \).

**Corollary 2.4.** Let \( m_n : S \to X, n \in N \), be an increasing sequence of \( \sigma \)-measures on \( S \) and let \( l : S \to Y \) be another \( \sigma \)-measure on \( S \). Suppose that \( \sup \{ m_n(A) : n \in N \} = m(A) \in X \), whenever \( A \in S \) and \( m_n \perp l \), for every \( n \in N \). Then \( m : S \to X \) is an \( \sigma \)-measure on \( S \) and \( m \perp l \).

3. **The Lebesgue Decomposition Theorem.** First we give the following:

**Lemma 3.1.** Let \( m : S \to X \) be an \( \sigma \)-measure on the \( \sigma \)-ring \( S \) and let \( \Lambda \) be a nonempty subfamily of \( S \) closed to countable unions. Suppose that \( X \) is a monotone complete of the countable type partially ordered semigroup. Then the function \( m_i : S \to X, m_i(A) = \sup \{ m(A \cap M) : M \in \Lambda \} \), is an \( \sigma \)-measure on \( S \) and for every \( A \in S \), there exists \( M \in \Lambda \) such that \( m_i(A) = m(A \cap M) \).

**Proof.** Let \( A \in S \). From the hypothesis it is easily verified that there exists an increasing sequence \( (M_n)_{n \in N} \) in \( \Lambda \) with \( M_n \uparrow M \in \Lambda \) and \( m_i(A \cap M) = \sup \{ m(A \cap M_n) : n \in N \} = m_i(A) \).

Next let \( (m_M)_{M \in \Lambda} \) be the increasing directed family of \( \sigma \)-measures on \( S \), such that \( m_M(A) = m(A \cap M) \) whenever \( M \in \Lambda \) and \( A \in S \). By Lemma 2.2 and from \( m_i(A) = \sup \{ m_M(A) : M \in \Lambda \}, A \in S \) it follows that \( m_i \) is an \( \sigma \)-measure on \( S \).

**Theorem 3.2 (Lebesgue Decomposition).** Let the \( \sigma \)-measures be \( m : S \to X, l : S \to Y \) on the \( \sigma \)-ring \( S \). Suppose, \( Y \) is a monotone complete of the countable type partially ordered semigroup. Then there exist unique \( \sigma \)-measures \( l_i : S \to Y, i = 1, 2, \) such that:
\[ l = l_1 + l_2, \quad l_1 \ll m, \quad l_2 \perp m, \quad l_1 \perp l_2. \]

**Proof.** We consider the functions: 
\[ l_i: S \to Y, \quad i = 1, 2, \quad l_1(A) = \sup\{l(A \cap M): M \in \Lambda\}, \quad l_2(A) = \sup\{l(A \cap Q): Q \in \Theta\} \] where \( \Lambda = \{M \in S: m(Q) = 0\} \) and \( \Theta = \{Q \in S: l_2(M) = 0\} \). Clearly from Lemma 3.1 the functions \( l_i: S \to Y, \quad i = 1, 2, \) are \( \sigma \)-measures on \( S \) and there exist \( M \in \Lambda, Q \in \Theta \) such that:

\[ l_1(A) = l(A \cap M) = l_1(A \cap M), \quad (4) \]
\[ l_2(A) = l(A \cap Q) = l_2(A \cap Q). \quad (5) \]

If \( m(A) = 0 \) then \( (A \cap M) \in \Theta \), hence \( l_1(A) = l(A \cap M) = l_2(A \cap M) = 0 \), namely \( l_1 \ll m \).

On the other hand \( (A - Q) \in \Lambda \) (because \( (A - Q) \notin \Lambda \) implies \( l_2(A - Q) > 0 \), so by (5) \( l_2(A) > l_2(A) \), that is a contradiction), therefore \( l(A - Q) = l_1(A - Q) = l_1(A) \). Thus \( l(A) = l(A - Q) + l(A \cap Q) = l_1(A) + l_2(A) \). Now from (4) and (5) one obviously has \( l_1 \perp l_2 \) and \( l_2 \perp m \). To show uniqueness let \( l = l_1 + l_2 = l_3 + l_4 \) be two such decompositions. Evidently \( l_4 \perp l_1 \) and \( l_2 \perp l_3 \). So from \( l_2 \leq l_3 + l_4 \) and \( l_4 \leq l_1 + l_2 \) imply \( l_2 \leq l_4 \) and \( l_4 \leq l_2 \), hence \( l_2 = l_4 \). Furthermore from \( l_1 \perp l_2, l_3 \perp l_2, l_1 \leq l_2 + l_3 \) and \( l_3 \leq l_1 + l_2 \) we also have \( l_1 \ll l_3 \).

4. Partially ordered topological semigroup-valued measures. Throughout this paragraph we suppose that \( X \) is a partially ordered topological semigroup, that is a partially ordered semigroup, equipped with a Hausdorff topology \( \tau_X \) such that the sets: 
\[ E_x := \{y \in X: y \geq x\}, \quad F_x := \{y \in X: y \leq x\} \text{ are } \tau_X\text{-closed,} \]
whenever \( x \in X \). In this place we give the well-known lemma.

**Lemma 4.1.** Let \( (x_i)_{i \in I} \) be an increasing directed family in the partially ordered topological semigroup \( X \) with \( \tau_X\lim x_i = x \) (convergence in the topology \( \tau_X \) of \( X \)). Then \( x = \sup\{x_i: i \in I\} \).

**Proof.** We set \( E_i = \{y \in X: y \geq x_i\} \) for every \( i \in I \), hence \( x \in \overline{E_i} = E_i \) (by \( \overline{E} \), we denote the closure of \( E_i \) in \( X \)), namely \( x \geq x_i \) for every \( i \in I \).

Moreover let \( z \) be an element of \( X \) so that:
\[ x_i \leq z, \quad \text{for any } i \in I. \]

Thus by the fact that the set \( F = \{y \in X: y \leq z\} \) is \( \tau_X\)-closed and hypothesis, one similarly has, \( x \in \overline{F} = F \), which proves the assertion. Next the topology \( \tau_X \) is called \( \sigma \)-compatible with the partial ordering if every majorised increasing sequence in \( X \) converges relative to the topology \( \tau_X \).

Now the function \( m: H \to X \) is a \( \tau_X \)-measure on the ring \( H \), if \( m \) is positive on \( H \) and \( m \) is \( \sigma \)-additive on \( H \) with respect to topological convergence in \( X \). The definitions and results of absolute continuity and singularity are similar as above.

In particular we obtain.

**Theorem 4.2.** Let the \( \tau_X \)-measure \( m: S \to X \) and the \( \tau_Y \)-measure \( l: S \to Y \) on the \( \sigma \)-ring \( S \). Suppose that \( Y \) is a monotone complete of the countable type
partially ordered topological semigroup and the topology $\tau_Y$ is $\sigma$-compatible with the partial ordering. Then there exist unique $\tau_Y$-measures $l_i: S \to Y$, $i = 1, 2$, such that:

$$l = l_1 + l_2, \quad l_1 \ll m, \quad l_2 \perp m, \quad l_1 \perp l_2.$$ 

The proof of the Theorem 4.2 follows from Lemma 4.1 and Theorem 3.2.

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