ON A. HURWITZ' METHOD IN ISOPERIMETRIC INEQUALITIES

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ABSTRACT. We show that if $M$ is complete simply connected with nonpositive sectional curvatures, $\Omega$ a minimal submanifold of $M$ with connected suitably oriented boundary $\Gamma$ then $\lambda^{1/2}V/A < (n - 1)^{1/2}/n$ where $V$ is the volume of $\Omega$, $A$ the volume of $\Gamma$, $\lambda$ the first nonzero eigenvalue of the Laplacian of $\Gamma$, and $n (> 2)$ is the dimension of $\Omega$.

In this note we prove the following extension of A. Hurwitz' [7] argument in his proof of the classical isoperimetric inequality.

THEOREM 1. Let $M$ be an $m$-dimensional complete simply connected Riemannian manifold all of whose sectional curvatures are nonpositive. Let $\Omega$ be an $n$-dimensional, $n \geq 2$, submanifold of $M$ with suitably oriented and connected boundary $\Gamma$. If $n < m$ assume $\Omega$ is minimal in $M$, i.e., its mean curvature vector vanishes identically. Let $V$ denote the $n$-volume of $\Omega$, $A$ the $(n - 1)$-volume of $\Gamma$, and $\lambda$ the first nonzero eigenvalue of the Laplacian acting on functions on $\Gamma$. Then

$$\sqrt{\lambda} \frac{V}{A} < \frac{\sqrt{n - 1}}{n}. \tag{1}$$

Clearly the inequality (1) is sharp, since any $n$-disk in $\mathbb{R}^m$ will yield equality in (1). We shall present a characterization of equality in (1) for two cases only.

THEOREM 2. If $n = m$ then equality in (1) implies that $\Omega$ is isometric to a disk in $\mathbb{R}^n$ endowed with the usual flat metric.

If $n < m$ and $M$ is $\mathbb{R}^m$ endowed with the usual flat metric then equality in (1) implies that $\Omega$ is the intersection of a disk in $\mathbb{R}^m$ with an $n$-dimensional affine space.

In particular, Theorems 1 and 2 recapture T. Carleman's theorem [2] (also cf. [3]), that for a minimal surface $\Omega$ in $\mathbb{R}^3$ with boundary $\Gamma$ consisting of a smooth Jordan curve, we have the inequality

$$L^2 - 4\pi A > 0 \tag{2}$$

where, just for the moment, $A$ is the area of $\Omega$ and $L$ the length of $\Gamma$. The
point is that since $\Gamma$ is 1-dimensional Wirtinger’s inequality implies $\lambda = 4\pi^2 / L^2$. Equality in (2) is achieved if and only if $\Omega$ is a flat disk in $R^3$.

We note that the first to free Carleman’s result and its generalizations from considerations of complex function theory was W. T. Reid [9], who proved the result in 3-space via J. H. Jellet’s formula (cf. (4) below), and C. C. Hsiung [6] who generalized Reid’s argument to surfaces in $n$-space. We wish to thank R. Osserman for helpful conversations, guesses, and references. We mention R. C. Reilly [10, esp. Corollary 1] has also obtained results using A. Hurwitz’ method as the starting point. Finally we refer the reader to the inequalities of D. Hoffman and J. Spruck [5] which, although not sharp, relate the volume and area alone.

1. The inequality. The manifolds $M$, $\Omega$, $\Gamma$ will be $C^\infty$, as will be the Riemannian metric under consideration. In general we will use differentiable for $C^\infty$.

We require some notation. Let $TM$ denote the tangent bundle of $M$, and let $\nabla$ denote covariant differentiation in $M$. For tangent vectors $\xi$, $\eta$ in the same fiber in $TM$ we denote their inner product by $\langle \xi, \eta \rangle$ and the associated norm of $\xi$, by $|\xi|$.

For an open set $U$ in $\Omega$ with coordinates chart $x: U \rightarrow R^n$ we let $\{\partial_1^x, \ldots, \partial_n^x\}$ denote the natural basis of tangent spaces to $\Omega$ at points of $U$ associated with $x$. We set

$$h_{jk} = \langle \partial_j^x, \partial_k^x \rangle, \quad \mathcal{C} = (h_{jk}), \quad \mathcal{C}^{-1} = (h^{jk}),$$

$j, k = 1, \ldots, n$. The mean curvature vector of $\Omega$ in $M$ will be denoted by $H$ (if $n = m$ then $H = 0$ by definition), $dV$ will denote the volume element of $\Omega$, and $dA$ that of $\Gamma$.

For a differentiable vector field $\xi: \Omega \rightarrow TM$ on $\Omega$ we let $\xi^\Gamma$ denote the projection of $\xi$ onto $T\Omega$. For any open neighborhood $U$ in $\Omega$ with coordinate chart $x$ as above we define the divergence of $\xi$, $\text{div}_\Omega \xi$, by

$$\text{div}_\Omega \xi = \sum_{j,k=1}^n h^{jk} \langle \nabla \partial_j^x \xi, \partial_k^x \rangle.$$ 

It is standard that the above definition is independent of the coordinate chart on $\Omega$. Furthermore the infinitesimal version of the first variation of area reads as [5, p. 719]

$$\text{div}_\Omega \xi^T = \text{div}_\Omega \xi + \langle \xi, H \rangle. \quad (3)$$

Thus if $\nu$ denotes the outward normal vector field of $\Gamma$ with respect to $\Omega$ then

$$\int_{\Omega} \int_{\Gamma} (\text{div}_\Omega \xi + \langle \xi, H \rangle) dV = \int_{\Gamma} \langle \xi, \nu \rangle dA. \quad (4)$$

We note that if $M$ is $R^n$ with the flat metric and $X$ is the position vector of points in $R^n$ suitably identified with a tangent vector field on $R^n$ then for $\xi = X |\Omega$ we have $\text{div}_\Omega \xi = n$ on all of $\Omega$ and the resulting formula (4) is that of J. H. Jellet [8].
Next we let $p \in M$, $\{e_1, \ldots, e_m\}$ be an orthonormal basis of $M_p$ and $y: M \to \mathbb{R}^m$ the Riemannian normal coordinates on $M$ determined by $(p; e_1, \ldots, e_m)$. Our assumptions concerning $M$ imply by the Hadamard-Cartan theorem that $y$ is indeed defined on all of $M$ and is a diffeomorphism of $M$ onto $\mathbb{R}^m$. It is standard that geodesics emanating from $p$ map onto rays emanating from the origin of $\mathbb{R}^m$.

**Lemma 1.** We may choose $p$, $\{e_1, \ldots, e_m\}$ so that the respective coordinate functions $y^D: M \to \mathbb{R}$, $D = 1, 2, \ldots, m$, of $y: M \to \mathbb{R}^m$ satisfy

$$\int_M y^D \, dA = 0. \quad (5)$$

**Proof.** Our argument is an easy adaptation of one given by H. F. Weinberger [13, p. 635].

Parallel translate the frame $\{e_1, \ldots, e_m\}$ along every geodesic emanating from $p$ and thereby obtain a differentiable orthonormal frame field $\{E_1, \ldots, E_m\}$ on $M$. Let $y_q: M \to \mathbb{R}^m$ denote the Riemann normal coordinates of $M$ determined by $\{E_1, \ldots, E_m\}$ at $q$, and let $(y_q)^D$, $D = 1, \ldots, m$ be the coordinate functions of $y_q$. Then

$$Y(q) = \sum_{D=1}^m \left( \int_M (y_q)^D \, dA \right) E_D(q)$$

is a continuous vector field on $M$. If we restrict $Y$ to a geodesic disk $B$ containing $\Omega$ then the convexity of $B$ implies that on the boundary of $B$, $Y$ points into $B$. The Brouwer fixed point theorem then implies that $Y$ has zero on $B$.

So we may assume that $p$, $\{e_1, \ldots, e_m\}$ actually satisfies (5). Let $X$ be the vector field on $M$ given by

$$X = \sum_{D=1}^m y^D \partial_D^y$$

where $\{\partial_D^y, D = 1, \ldots, m\}$ is the natural basis of tangent spaces associated with the coordinate chart $y$. Of course if $M$ is $\mathbb{R}^m$ with its usual flat metric then $X$ is naturally identified with the position vector. Next set $\xi = X|\Omega$, i.e. the restriction of $X$ to $\Omega$. Then standard arguments using the Rauch comparison theorem [5, p. 721] imply

$$n \leq \text{div}_\Omega \xi. \quad (6)$$

Thus, if $H = 0$ on $\Omega$, then (4), (5), (6) imply

$$nV \leq \int_M \langle \xi, \nu \rangle \, dA \leq \int_M |\xi| \, dA$$

$$\leq A^{1/2} \left( \int_M |\xi|^2 \, dA \right)^{1/2} = A^{1/2} \left( \int_M \sum_D (y^D)^2 \, dA \right)^{1/2}$$

$$\leq (A/\lambda)^{1/2} \left( \int_M \sum_D |\text{grad}_T y^D|^2 \, dA \right)^{1/2}$$

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The expression \( \text{grad}_\Gamma y^D \) denotes the gradient of \( (y^D|\Gamma) \) in \( \Gamma \). The last inequality is, combined with (5), Lord Rayleigh's characterization of the first nonzero eigenvalue \( \lambda \) of the Laplacian on \( \Gamma \).

It remains to verify the estimate

\[
\sum_D |\text{grad}_\Gamma y^D|^2 \leq n - 1
\]

on all of \( \Gamma \).

Let \( q \in \Gamma, u: G \to R^{n-1} \) be a coordinate chart on \( \Gamma \) about \( q \), \( \{\partial_u^\alpha, \alpha = 1, \ldots, n - 1\} \) the natural basis of tangent spaces to \( \Gamma \) at points of \( G \) such that \( \{\partial_u^1, \ldots, \partial_u^{n-1}\} \) is orthonormal at \( q \). For \( y: M \to R^m \) let \( g_{AB} = \langle \partial_u^A, \partial_u^B \rangle \); then the Rauch comparison theorem implies that the eigenvalues of \( (g_{AB}) \) are all \( > 1 \). Thus at \( q \)

\[
\sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{m} g_{\alpha\beta} \frac{\partial y}{\partial u^\alpha} \frac{\partial y}{\partial u^\beta} \leq \sum_{\alpha=1}^{n-1} \sum_{\beta=1}^{m} \delta_{\alpha\beta} \frac{\partial y}{\partial u^\alpha} \frac{\partial y}{\partial u^\beta} = n - 1,
\]

and Theorem 1 is proven.

2. The case of equality for \( m = n \). If \( m = n \) and we have equality in (1) then \( |\xi| \) is constant on \( \Gamma \) from which one concludes that \( \Gamma \) is a geodesic sphere bounding the geodesic disk \( \Omega \). Furthermore \( \text{div}_\Gamma \xi = n \) on all of \( \Omega \). But this in turn implies by [1, pp. 253–257; 5, ibid.] that \( \Omega \) is isometric to a disk in \( R^n \).

3. The case of equality for \( n < m, M = R^m \). Since \( M = R^m \) we identify \( \xi \) at \( q \) with the position vector at \( q \), and note that equality in (1) implies that \( \Gamma \) is contained in some sphere \( S^{m-1}(\rho) \) of radius \( \rho \) about the origin and that, furthermore, that the outward normal, \( v \), of \( \Gamma \) with respect to \( \Omega \) is \( \rho^{-1}\xi \).

Next we note that equality in (1) implies, by the Rayleigh characterization of eigenvalues, that the functions \( y^D|\Gamma \) are eigenvalues of \( \Delta_\Gamma \), the Laplacian of \( \Gamma \), with eigenvalue \( \lambda \), for \( D = 1, \ldots, m \). We therefore have

\[
0 = (1/2)\Delta_\Gamma \sum_D (y^D|\Gamma)^2 = (1/2) \sum_D \Delta_\Gamma (y^D|\Gamma)^2
\]

\[
= \sum_D \left\{ (y^D|\Gamma)\Delta_\Gamma (y^D|\Gamma) + |\text{grad}_\Gamma (y^D|\Gamma)|^2 \right\} = -\lambda \rho^2 + (n - 1);
\]

thus \( \lambda = (n - 1)/\rho^2 \) and by T. Takahashi's theorem [12] \( \Gamma \) is minimal in \( S^{m-1}(\rho) \).

It is standard that if \( \Gamma \) is minimal in \( S^{m-1}(\rho) \) then the cone over \( \Gamma \) through the origin is minimal in \( R^m \) (cf. e.g., [11, p. 97]). However \( \Omega \) and the cone over \( \Gamma \) are tangent along \( \Gamma \). By the Cauchy-Kowalewsky theorem [4, p. 39] \( \Omega \) is the cone over \( \Gamma \). Since \( \Omega \) is assumed to have no singularities, the cone must therefore be an \( n \)-dimensional subspace of \( R^m \). Thus Theorem 2 is proven.

Remark. The differentiability of \( \Omega \) is crucial. Indeed given any minimal
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submanifold $\Gamma$ of $S^{n-1}_\rho$, let $\Omega$ be the cone over $\Gamma$ through the origin of $R^m$. One easily sees that equality is obtained in (1).

REFERENCES


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