A NOTE ON STABLE EQUIVALENCE AND
NAKAYAMA ALGEBRAS

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Abstract. Two artin algebras $A$ and $A'$ are said to be stably equivalent if
the categories of finitely generated modules modulo projective for $A$ and $A'$
are equivalent categories.

If $A'$ is stably equivalent to a Nakayama (i.e. generalized uniserial)
algebra $A$, we prove that $A$ and $A'$ have the same number of nonprojective
simple modules. And if $A$ and $A'$ are stably equivalent indecomposable
Nakayama algebras where each indecomposable projective module has
length at least 3, then $A$ and $A'$ have the same admissible sequences.

Introduction. We recall that two artin algebras $A$ and $A'$ are said to be
stably equivalent if the categories $\text{mod } A$ and $\text{mod } A'$ modulo projectives are
equivalent categories, where $\text{mod } A$ denotes the category of finitely generated
left $A$-modules.

An open problem about stable equivalence, which is of importance in the
modular representation theory of finite groups, is whether two stably equiva-

lent algebras $A$ and $A'$ have the same number of nonprojective simple
modules. In §1 we show that if one of the algebras is a Nakayama (i.e.
generalized uniserial) algebra, then this is the case. It is also known to be true
if $A$ is hereditary [2, Theorem 5.5] and if $A$ has no nonzero projective injective
modules [1, Proposition 3.6]. See also [11].

In §2 we prove the result, announced in [10], that if $A$ and $A'$ are stably
equivalent Nakayama algebras such that each indecomposable projective
module has length at least 3, then $A$ and $A'$ have the same admissible
sequences (see §2 for definition). Hence if they are both basic algebras over
an algebraically closed field, they must be isomorphic [8, Satz 7]. Our method
is based upon using the notion of irreducible map introduced in [3].

In connection with the above, we point out that if $A'$ is an artin algebra
stably equivalent to a Nakayama algebra $A$, then $A'$ is not necessarily
Nakayama [9]. If, however, $A$ has Loewy length at most 3, then $A'$ is
Nakayama [9, Corollary 2.2]. However, there are non Nakayama algebras $A'$
stably equivalent to Nakayama algebras of Loewy length 4, and in [12] we
give, as announced in [10], a characterization of the artin algebras stably
equivalent to Nakayama algebras of Loewy length 4. P. Gabriel and C.
Riedtmann have studied artin algebras stably equivalent to self-injective algebras over an algebraically closed field. Stable equivalence of Nakayama algebras has also been studied in [6].

1. Let \( \Lambda \) be an artin algebra, and denote by \( \text{mod} \, \Lambda \) the category of finitely generated left \( \Lambda \)-modules. Denote by \( \text{mod} \, \Lambda \) the category \( \text{mod} \, \Lambda \) modulo projectives, that is, the objects of \( \text{mod} \, \Lambda \) are the same as those of \( \text{mod} \, \Lambda \), now denoted \( M \) for \( M \) in \( \text{mod} \, \Lambda \), and \( \text{Hom}(M, N) = \text{Hom}_\Lambda(M, N) / P(M, N) \), where \( P(M, N) \) consists of the maps from \( M \) to \( N \) which factor through a projective \( \Lambda \)-module. For \( f: M \to N \), let \( \bar{f} \) denote the image of \( f \) in \( \text{Hom}(M, N) \). \( \Lambda \) and \( \Lambda' \) are said to be stably equivalent if there is an equivalence of categories \( \alpha: \text{mod} \, \Lambda \to \text{mod} \, \Lambda' \). \( \alpha \) induces a correspondence, also denoted by \( \alpha \), between the objects in \( \text{mod} \, \Lambda \) and in \( \text{mod} \, \Lambda' \), the categories of finitely generated modules with no nonzero projective summands.

The following lemmas about stable equivalence will be useful in what follows.

**Lemma 1.1.** Let \( \Lambda \) be an artin algebra and \( f: A \to B \) an epimorphism in \( \text{mod} \, \Lambda \). If \( A \) and \( B \) have no nonzero projective summands, then \( \bar{f} \) is not zero.

**Proof.** If \( \bar{f} \) was zero, we would have some commutative diagram

\[
\begin{array}{ccc}
P & \to & P \\
\downarrow s & & \downarrow t \\
A & \to & B \\
\end{array}
\]

where \( P \) is projective. Since \( A \) has no nonzero projective summand \( \text{Im} \, s \) must be contained in \( rP \), where \( r \) denotes the radical of \( \Lambda \). Hence \( \text{Im} \, f \subseteq rB \), which contradicts the fact that \( f \) is an epimorphism. It follows that \( \bar{f} \) is not zero.

**Lemma 1.2.** Let \( \Lambda \) be a Nakayama algebra and \( f: A \to B \) a monomorphism. If \( A \) and \( B \) are indecomposable nonprojective \( \Lambda \)-modules, then \( \bar{f} \) is not zero.

**Proof.** Assume that \( \bar{f} \) is zero. We then have some commutative diagram

\[
\begin{array}{ccc}
P & \to & P \\
\downarrow s & & \downarrow t \\
A & \to & B \\
\end{array}
\]

where \( t: P \to B \) is a projective cover. Since \( P \) is indecomposable, \( \text{soc} \, P \) is simple, and since \( t: P \to B \) is not a monomorphism, \( t(\text{soc} \, P) \) must be zero. It follows that \( f(\text{soc} \, A) = ts(\text{soc} \, A) \) is zero, since \( s(\text{soc} \, A) \subseteq \text{soc} \, P \). This contradicts the fact that \( f \) is a monomorphism. Hence it follows that \( \bar{f} \) is zero.

In this section we prove the following result, which under the assumption that \( \Lambda \) and \( \Lambda' \) are self-injective, is essentially proved in [7].

**Theorem 1.3.** Let \( \Lambda \) and \( \Lambda' \) be stably equivalent artin algebras. If \( \Lambda \) is
Nakayama, then $\Lambda$ and $\Lambda'$ have the same number of nonprojective simple modules.

**Proof.** Let $\alpha : \text{mod } \Lambda' \rightarrow \text{mod } \Lambda$ be an equivalence, where $\Lambda$ is Nakayama. Let $S_1, \ldots, S_n$ be the nonprojective simple $\Lambda'$-modules. We then claim that $\text{soc}(\alpha S_i), \ldots, \text{soc}(\alpha S_n)$ are nonisomorphic simple $\Lambda$-modules. Since $\Lambda$ is Nakayama, and the $\alpha S_j$ are all indecomposable, we know that the $\text{soc}(\alpha S_j)$ are all simple. Assume that for $i \neq j$, $\text{soc}(\alpha S_i) \cong \text{soc}(\alpha S_j)$. We can assume without loss of generality that the length of $\alpha S_i$ is at most equal to the length of $\alpha S_j$. Since $\Lambda$ is Nakayama, we know by the structure of indecomposable $\Lambda$-modules that there is a monomorphism $\iota : \alpha S_i \rightarrow \alpha S_j$. Since $\text{Hom}(S_i, S_j) = 0$, we have $\text{Hom}(\alpha S_i, \alpha S_j) = 0$, which is a contradiction by Lemma 1.2. This shows that the $\text{soc}(\alpha S_i)$ are all nonisomorphic.

We further observe that since $\text{soc}(\alpha S_i)$ is a composition factor of some $P_i/\text{soc } P_i$, where $P_i$ is indecomposable projective, $\text{soc}(\alpha S_i)$ is not a simple projective $\Lambda$-module. Hence $\Lambda$ has at least $n$ nonisomorphic simple nonprojective modules.

Let now $T$ be a simple nonprojective $\Lambda$-module, and let $A$ be the indecomposable nonprojective $\Lambda'$-module such that $\alpha A = T$. Then there is an epimorphism $\iota : A \rightarrow S_i$ for some $i$, hence we know by Lemma 1.1 that $\text{Hom}(A, S_i)$ is not zero. It then follows that $\text{Hom}(T, \alpha S_i)$ is not zero. This means that there is a nonzero map, hence a monomorphism, from $T$ to $\alpha S_i$, so that $T \cong \text{soc } \alpha S_i$. This shows that $\Lambda$ has exactly $n$ simple nonprojective modules, and the proof is done.

2. Let $\Lambda$ be a Nakayama algebra. We shall recall some well known facts about indecomposable Nakayama algebras (see [8], [5]). There is an ordering of the indecomposable projective $\Lambda$-modules $P_1, \ldots, P_n$, such that $P_i/\text{soc } P_i \cong P_{i+1}/\text{soc } P_{i+1}$, $i = 1, \ldots, n - 1$, and either $P_n/\text{soc } P_n \cong P_{i+1}/\text{soc } P_{i+1}$. The sequence of integers $c_1, \ldots, c_n$, where $c_i$ is the length of $P_i$, is called an admissible sequence for $\Lambda$. An admissible sequence is determined up to cyclic permutation. If $\Lambda$ is a basic indecomposable Nakayama algebra over an algebraically closed field $k$, then the associated admissible sequence uniquely determines $\Lambda$ [8]. We recall that we showed in [9] that if $\Lambda$ and $\Lambda'$ are stably equivalent indecomposable Nakayama algebras, with no simple projective modules then the associated admissible sequences have the same length, that is, the same number of $c_i$ (this also follows from Theorem 1.1, since it says that $\Lambda$ and $\Lambda'$ have the same number of nonprojective simple modules), and that the number of $c_i$ equal to a given number $t \geq 2$ is the same for both admissible sequences. (See [9, Proposition 1.1].) In the special case that all $c_i$ are 3 or 4, we showed that the admissible sequences are actually the same ([9, Proposition 3.1]. [10, Proposition 3.1]). Here we shall, as announced in [10], prove the same thing just assuming that all $c_i$ are at least 3. We give an example to show that the result is not true in general.

To do this we shall use the notion of irreducible maps introduced in [3]. We recall that for an artin algebra $\Lambda$, a map $f : A \rightarrow B$ in mod $\Lambda$ is said to be
irreducible if \( f \) is neither a split epimorphism nor a split monomorphism, and given any commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{h} \\
X & \xrightarrow{f} & h
\end{array}
\]

then either \( g \) is a split monomorphism or \( h \) is a split epimorphism. It is not hard to see that if \( \Lambda \) is a Nakayama algebra, then a map \( f: A \to B \) between indecomposable modules \( A \) and \( B \) in \( \text{mod}\, \Lambda \) is irreducible if and only if \( f \) is a monomorphism and \( l(A) = l(B) - 1 \) or \( f \) is an epimorphism and \( l(A) = l(B) + 1 \) (see [3]). Here \( l \) denotes length. Another useful fact about irreducible maps that we shall need here is that if \( \alpha: \text{mod}\, \Lambda \to \text{mod}\, \Lambda' \) is an equivalence and \( f: A \to B \) is an irreducible map where \( A \) and \( B \) are in \( \text{mod}_p\Lambda \), then \( f' : A' \to B' \), where \( A' = \alpha(A) \), \( B' = \alpha(B) \), \( \alpha(f) = f' \), is also irreducible [4, Proposition 12].

To prove that two stably equivalent indecomposable Nakayama algebras, where each indecomposable projective module has length at least 3, have the same admissible sequences, we shall need a few lemmas.

**Lemma 2.1.** Let \( \Lambda \) be a Nakayama algebra.

(a) Let \( S_1 \) be a simple \( \Lambda \)-module and assume that the projective cover \( P_1 \) of \( S_1 \) has length at least 3. Then \( S_2 = \tau P_1 / \tau^2 P_1 \) is the unique indecomposable nonprojective \( \Lambda \)-module with the property that there is some chain \( S_2 \xrightarrow{f} X \xrightarrow{g} S_1 \) of irreducible maps between indecomposable nonprojective \( \Lambda \)-modules such that \( gf \) is zero.

(b) Let \( P_1 \) be an indecomposable projective \( \Lambda \)-module of length \( n > 3 \). Then there is a unique indecomposable nonprojective \( \Lambda \)-module \( A \) such that there is some chain \( A \xrightarrow{f} Y \xrightarrow{g} P_1 / \tau^{n-1} P_1 \) of irreducible maps between indecomposable nonprojective modules such that \( gf \) is zero, if and only if \( \tau P_1 \) is not projective.

If \( \tau P_1 \) is not projective, then \( A \equiv \tau P_1 \).

**Proof.** (a) \( P_1 / \tau^2 P_1 \) is the only indecomposable (nonprojective) \( \Lambda \)-module such that there is some irreducible map \( P_1 / \tau^2 P_1 \to S_1 \). The only indecomposable \( \Lambda \)-modules having irreducible maps to \( P_1 / \tau^2 P_1 \) are \( P_1 / \tau^3 P_1 \) and \( \tau P_1 / \tau^2 P_1 = S_2 \). \( S_2 \) is not projective, and for any chain

\[
S_2 \xrightarrow{f} P_1 / \tau^2 P_1 \xrightarrow{g} S_1
\]

of irreducible maps, \( gf \) is clearly zero. And if \( P_1 / \tau^3 P_1 \) is not projective, and \( P_1 / \tau^3 P_1 \xrightarrow{f} P_1 / \tau^2 P_1 \xrightarrow{g} S_1 \) is a chain of irreducible maps, then \( gf \) is an epimorphism, so that \( gf \neq 0 \) by Lemma 1.1.

(b) \( \tau P_1 / \tau^{n-1} P_1 \) is the only indecomposable nonprojective \( \Lambda \)-module with an irreducible map to \( P_1 / \tau^{n-1} P_1 \). The indecomposable nonprojective modules with an irreducible map to \( \tau P_1 / \tau^{n-1} P_1 \) are \( \tau^2 P_1 / \tau^{n-1} P_1 \), if \( n > 4 \), and \( \tau P_1 \), if \( \tau P_1 \) is not projective. For \( n > 4 \), any composition of irreducible maps
is a monomorphism. Hence $gf$ is not zero by Lemma 1.2. If $rP_1$ is not projective, let $rP_1 \xrightarrow{f} rP_1/r^\sigma P_1 \xrightarrow{g} P_1/r^\sigma P_1$ be a chain of irreducible maps between indecomposable nonprojective $\Lambda$-modules. It is then easy to see that $gf = 0$. This finishes the proof of (b).

**Lemma 2.2.** Let $\alpha: \text{mod } \Lambda \rightarrow \text{mod } \Lambda'$ be a stable equivalence between Nakayama algebras, and let $S_1$ be a simple $\Lambda$-module such that $L(P_1) = n_1 \geq 3$, where $P_1$ is a projective cover for $S_1$. Assume that $\alpha S_1 \cong T_1$, where $T_1$ is a simple $\Lambda'$-module, and denote by $Q_1$ a projective cover of $T_1$. Then $\alpha S_2 \cong T_2$, where $S_2 = rP_1/r^2P_1$ and $T_2 = rQ_1/r^2Q_1$.

**Proof.** We have the following chains of irreducible maps between indecomposable nonprojective $\Lambda'$-modules.

$$\alpha S_2 \xrightarrow{f} \alpha \left( P_1/r^2P_1 \right) \xrightarrow{g} \alpha S_1 = T_1 \quad \text{and} \quad T_2 \xrightarrow{f'} Q_1/r^2Q_1 \xrightarrow{g'} T_1.$$

To get the first chain, we first consider $S_2 \rightarrow P_1/r^2P_1 \rightarrow S_1$, and use the connection between stable equivalence and irreducible maps. Then we have $gf = 0$, and also $g'f' = 0$. By Lemma 2.1(a) we can then conclude that $\alpha S_2 \cong T_2$.

**Lemma 2.3.** Let $\alpha: \text{mod } \Lambda \rightarrow \text{mod } \Lambda'$ be a stable equivalence between indecomposable Nakayama algebras, and assume that each indecomposable projective $\Lambda$-module and $\Lambda'$-module has length at least 3. If for some simple $\Lambda$-module $S_1$, $\alpha S_1$ is not simple, then the admissible sequences for $\Lambda$ and $\Lambda'$ are both constant, and are hence the same.

**Proof.** Assume that $\alpha S_1$ is not simple. $(,S_1)$ denotes the functor from mod $\Lambda$ to Ab given by $(,S_1)(A) = \text{Hom}(A, S_1)$. By [9, Proposition 1.1] we know that $(,S_1)$ is uniserial and that $\alpha S_1 \cong Q_1/r^\sigma Q_1$, where $Q_1$ is an indecomposable projective $\Lambda'$-module of length $n \geq 3$. We further know that

$$l(P_1) - 1 = l((,S_1)) = l((,Q_1/r^\sigma Q_1)) = l(Q_1) - 1,$$

where $P_1$ denotes a projective cover for $S_1$. The chain $S_2 \xrightarrow{f} P_1/r^2P_1 \xrightarrow{g} S_1$ of irreducible maps gives rise to a chain of irreducible maps $\alpha S_2 \xrightarrow{f'} \alpha(P_1/r^2P_1) \xrightarrow{g'} \alpha S_1$, where $gf$ is zero. By Lemma 2.1, $rQ_1$ is not projective, and $\alpha S_2 \cong rQ_1$. Since $rQ_1$ cannot be simple, we have as above that $rQ_1 \cong Q_2/r^m Q_2$, where $Q_2$ is an indecomposable projective $\Lambda'$-module of length $m \geq 3$, and also that

$$l(P_2) - 1 = l((,S_2)) = l((,Q_2/r^m Q_2)) = l(Q_2) - 1.$$

Here $P_2$ denotes a projective cover for $S_2$. Since clearly $l(Q_1) = l(Q_2)$, we have by the above that $l(P_1) = l(P_2)$. The proof of the lemma is then easily finished.

We are now ready to prove our desired result.

**Theorem 2.4.** Let $\Lambda$ and $\Lambda'$ be stably equivalent indecomposable Nakayama
algebras where each indecomposable projective module has length at least 3. Then \( \Lambda \) and \( \Lambda' \) have the same admissible sequences.

**Proof.** Let \( \alpha: \text{mod } \Lambda \to \text{mod } \Lambda' \) be an equivalence. Assume first that there is some simple \( \Lambda \)-module \( S \) such that \( \alpha S \) is not simple. Then we are done by Lemma 2.3. If for every simple \( \Lambda \)-module \( S \), \( \alpha S \) is simple, we are done using Lemma 2.2 since \( l(P) - 1 = l((,S)) \), where \( P \) is a projective cover for \( S \) [9]. This finishes the proof of the theorem.

We have only dealt with indecomposable Nakayama algebras. This is no loss of generality because if \( \Lambda \) and \( \Lambda' \) are stably equivalent Nakayama algebras where each indecomposable projective module has length at least 3, and \( \Lambda = \Lambda_1 \times \cdots \times \Lambda_n \) and \( \Lambda = \Lambda'_1 \times \cdots \times \Lambda'_m \) are decompositions into products of indecomposable algebras, then \( m = n \), and the numbering can be made such that \( \Lambda_i \) and \( \Lambda'_i \) are stably equivalent. This is a direct consequence of the following result.

**Proposition 2.5.** Let \( \Lambda \) be a Nakayama algebra such that each indecomposable projective \( \Lambda \)-module has length at least 3. Then \( \Lambda \) is indecomposable if and only if \( \text{mod } \Lambda \) is an indecomposable category.

**Proof.** Assume that \( \Lambda \) is indecomposable, and let \( S_1, \ldots, S_n \) be the simple \( \Lambda \)-modules, ordered in a way such that \( S_{i+1} \equiv rP_i/t^2P_i \), \( i = 1, \ldots, n - 1 \), \( S_1 \equiv rP_n/t^2P_n \). We have nonzero morphisms \( S_{i+1} \to P_i/t^2P_i \) and \( P_i/t^2P_i \to S_i \).

And if \( X \) is an arbitrary indecomposable nonprojective \( \Lambda \)-module, there is clearly some \( S_i \) such that \( \text{Hom}(X, S_i) \neq 0 \). From this we conclude that \( \text{mod } \Lambda \) is indecomposable.

If \( \text{mod } \Lambda \) is indecomposable and \( \Lambda \) has no semisimple ring summand, then \( \Lambda \) is clearly indecomposable.

We end this section by pointing out that if \( \Lambda \) and \( \Lambda' \) are stably equivalent indecomposable Nakayama algebras, then we cannot in general conclude that they have the same admissible sequences. For example, if \( k \) is a field, then \( k[x]/(x^2) \) and \( (k, 3) \) are stably equivalent Nakayama algebras with admissible sequences \( (2) \) and \( (2, 1) \) respectively.

**References**


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