K-THEORY AND K-HOMOLOGY RELATIVE TO A II\textsubscript{∞}-FACTOR

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Abstract. Let X be a compact space and M be a factor of type II\textsubscript{∞} acting on a separable Hilbert space. Let \( K_M(X) \) denote the Grothendieck group generated by the semigroup of isomorphism classes of \( M \)-vector bundles over \( X \), and, if \( X \) is also metric, let \( \text{Ext}^M(X) \) denote the group of equivalence classes of extensions of \( C(X) \) relative to \( M \). We show that \( K_M(X) \) is the direct sum of the even-dimensional Čech cohomology groups of \( X \), and that \( \text{Ext}^M(X) \) is the direct product of the odd-dimensional Čech homology groups of \( X \).

Introduction. Recently Brown, Douglas, and Fillmore [8] have constructed a generalised homology theory called K-homology, which, in a sense made rigorous in [8], is dual to K-theory. Their construction is in terms of extensions of commutative C*-algebras by the ideal of compact operators on a separable Hilbert space. Fillmore [12] and Cho [9] have investigated the analogous construction with the compact operators replaced by the closed two-sided ideal generated by the finite projections in a factor of type II\textsubscript{∞}. They have constructed (see [9]) a generalised homology theory \( \{\text{Ext}^M_n\} \) on the category of compact metric spaces, which we shall call K-homology relative to the II\textsubscript{∞}-factor \( M \). In [6] Breuer has considered a theory of vector bundles relative to \( M \) and introduced a functor \( K_M \) which has topological properties like those of K-theory. We shall construct a generalised cohomology theory \( \{K^n_M\} \) (K-theory relative to \( M \)) from Breuer’s functor, identify it in terms of the conventional K-functor and show that \( K_M(X) \) is the direct sum of the even-dimensional real cohomology of \( X \) for any compact space \( X \). Then we shall deduce the corresponding result for \( \text{Ext}^M \); namely that \( \text{Ext}^M(X) \) is the direct product of the odd-dimensional real homology of \( X \). We mention that the results in this note all follow in standard fashion from the recent literature; our goal is merely to point out some interesting consequences of the work of Breuer [6] and Cho [9]. Along the way we provide a proof of Proposition 2, which has been stated and used by Singer in [18].

First we set up some notation. Throughout, all topological spaces will be Hausdorff, and \( M \) will be a factor of type II\textsubscript{∞} acting on a separable Hilbert space \( H \). We shall denote by \( P_f(M) \) the set of finite projections of \( M \) and by \( \dim: P_f(M) \rightarrow \mathbb{R}^+ \) the Murray-von Neumann dimension function of \( M \). For

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details on such matters, we refer to [10]. In addition, we shall write $\mathcal{K}(M)$ for the closed two-sided ideal of $M$ generated by $P_j(M)$, $\mathfrak{H}(M)$ for the quotient algebra $M/\mathcal{K}(M)$ and $\mathfrak{S}(M)$ for the set of operators which are Fredholm relative to $M$ (cf. [5]). Our terminology as regards $K$-theory will be that of [1]. By a generalised (Čech) cohomology theory on compact pairs, we shall mean a sequence $\{K^n\}$ of contravariant functors which satisfy the three axioms of continuity, excision and exactness (cf. [20, §1]). We observe that continuous functors are necessarily homotopy invariant [20, Theorem 2.1], so that such theories satisfy the first six of the Eilenberg-Steenrod axioms. We shall need the following lemma.

**Lemma.** If $\mu: \{H^n\} \to \{K^n\}$ is a natural transformation between generalised cohomology theories such that $\mu: H^n(X) \to K^n(X)$ is an isomorphism for all $n$ when $X$ is a point, then $\mu$ is a natural equivalence.

**Proof.** That $\mu$ is an equivalence on compact polyhedra follows from the argument of [19, Theorem 4.8.10]. But every compact space is the inverse limit of spaces with the homotopy type of compact polyhedra [19, Lemma 6.6.7], and so the result holds on the category of compact spaces.

1. Let $X$ be a compact space. Breuer [6] introduced the notion of an $M$-vector bundle over $X$—namely, a Hilbert space bundle over $X$ whose transition functions take values in $M$ and whose fibres are of the form $E(H)$ for some $E$ belonging to $P_j(M)$. The set $\text{Vect}_M(X)$ of $M$-isomorphism classes of $M$-vector bundles over $X$ is a semigroup under direct sum; if $f$ is a continuous map from $Y$ to $X$, then $f$ induces (via pull-back of bundles) a semigroup homomorphism $f^*: \text{Vect}_M(X) \to \text{Vect}_M(Y)$. If we denote the Grothendieck group of $\text{Vect}_M(X)$ by $K_M(X)$, then $K_M$ is a contravariant functor from compact spaces to abelian groups. If $X$ is a compact space with distinguished base point $x_0$, and $i: \{x_0\} \to X$ is the inclusion, then we write $\tilde{K}_M(X)$ for the kernel of the map $i^*: K_M(X) \to K_M(\{x_0\})$. Breuer proved that $K_M$ is homotopy invariant, and that $K_M(X)$ is a module over the ring $K(X)$; it is easy to check from the definition [6, p. 417] that this module action is natural. The main result of Breuer’s article is the periodicity theorem for $K_M$; namely that for any locally compact space $X$, $K_M(\mathbb{R}^2 \times X) \cong K_M(X)$, where for $Y$ locally compact $K_M(Y)$ stands for the reduced group $\tilde{K}_M(Y \cup \{\infty\})$ of the one point compactification of $Y$. This isomorphism is natural since the inverse $\beta_X$ is defined in terms of the module action.

We define $K_M^n(X) = K_M(\mathbb{R}^n \times X)$ (for $n > 0$) and, inductively, $K_M^n(X) = K_M^{n-2}(X)$ for positive $n$. If for a compact pair $(X, Y)$ we now set $K_M^n(X, Y) = \tilde{K}_M^n(X/Y)$ (the base point is $Y/Y$) then $\{K_M^n\}$ is a sequence of contravariant functors from compact pairs to abelian groups.

**Proposition 1.** $\{K_M^n\}$ is a generalised cohomology theory on compact pairs.

**Proof.** That $\{K_M^n\}$ satisfies excision is obvious. To verify continuity and exactness we shall use the theorem of Breuer that $K_M(X) \cong [X, \mathfrak{S}(M)]$ (see
(6, Theorem 1, p. 414)); an inspection of Breuer’s construction yields that the isomorphism is natural. Since \( \mathcal{T}(M) \) is an open set in the Banach space \( M \) ([5, II, Corollary 2 to Theorem 1]), \( \mathcal{T}(M) \) and its loop spaces \( \Omega^n \mathcal{T}(M) \) are ANR’s (cf. [14, Chapter 1]). It follows from the periodicity theorem that \( \pi_n(\mathcal{T}(M)) = \pi_n(\Omega^n \mathcal{T}(M)) \) for every \( n > 0 \), and so \( \mathcal{T}(M) \) and \( \Omega^n \mathcal{T}(M) \) are homotopy equivalent by [17, Theorem 15]. Thus \( \{ K^n_M \} \) is given by a spectrum, and so by [21, §5] satisfies the exactness axiom on finite complexes. We can deduce that \( K_M \) is continuous from the fact that \( \mathcal{T}(M) \) is an ANR, and the result follows.

If \( X \) is a compact space, \( r \in \mathbb{R}^+ \) and \( E \in P_f(M) \) satisfies \( \dim E = r \), then, as in the construction of the module action ([6, p. 417]), there is a map \( \lambda: \text{Vect}(X) \to \text{Vect}_M(X) \) given by \( \lambda(a) = a \otimes (X \times E(H)) \). Thus there is a pairing \( (a, r) \to \lambda(a): \text{Vect}(X) \otimes \mathbb{R}^+ \to \text{Vect}_M(X) \) which induces a natural transformation \( \lambda: K(\cdot) \otimes \mathbb{Z} \to K_M(\cdot) \). We observe that \( \lambda: K(X) \otimes \mathbb{R} \to K_M(X) \) is an isomorphism when \( X \) is a one point space.

**Proposition 2 (Singer).** The functors \( K(\cdot) \otimes \mathbb{Z} \mathbb{R} \) and \( K_M(\cdot) \) are naturally equivalent (via \( \lambda \)) on the category of compact spaces. In particular, \( K_M(\cdot) \) is independent of the factor \( M \).

**Proof.** The functors \( K^* \) form a generalised cohomology theory, and this implies that \( K^*(\cdot) \otimes \mathbb{Z} \mathbb{R} \) do also. For clearly \( K^n(\cdot) \otimes \mathbb{R} \) is a sequence of contravariant functors satisfying the excision axiom; the exactness axiom for \( K(\cdot) \otimes \mathbb{R} \) follows since \( \mathbb{R} \) is torsion-free and the continuity axiom follows since tensoring with \( \mathbb{R} \) commutes with direct limits ([3, pp. 33–34]). Let \( X \) be a compact space and let \( \text{Per}: K(X) \to K(\mathbb{R}^2 \times X) \) and \( \text{Per}_M: K_M(X) \to K_M(\mathbb{R}^2 \times X) \) denote the periodicity maps of \( K \)-theory and \( K_M \)-theory respectively. Then \( \text{Per} \) is given by taking the external product with the Bott element \( b \in K(\mathbb{R}^2) \) ([4, p. 118]), and \( \text{Per}_M \) is the analogous external product for \( K_M \)-theory with the same element \( b \in K(\mathbb{R}^2) \) ([6, p. 426]). It follows from elementary properties of the external product (cf. [6, §4.11]) that the diagram

\[
\begin{align*}
K(X) \otimes \mathbb{R} & \xrightarrow{\text{Per} \otimes \text{id}} K(\mathbb{R}^2 \times X) \otimes \mathbb{R} \\
\downarrow \lambda & \quad \downarrow \lambda \\
K_M(X) & \xrightarrow{\text{Per}_M} K_M(\mathbb{R}^2 \times X)
\end{align*}
\]

commutes. Hence \( \lambda \) can be extended to give a natural transformation between the generalised cohomology theories \( K^*(\cdot) \otimes \mathbb{R} \) and \( K_M^*(\cdot) \). As observed above \( \lambda: K^n(\text{pt}) \otimes \mathbb{R} \to K_M^n(\text{pt}) \) is an isomorphism when \( n = 0 \); since every \( M \)-vector bundle on \( S^1 \) is trivial ([6, Corollary 2, p. 404]) it is also an isomorphism for \( n = -1 \), and it follows that \( \lambda \) is an isomorphism for all \( n \in \mathbb{Z} \). The results now follow from the lemma in the introduction.

It is a standard result in \( K \)-theory that for a compact space \( X \), \( K(X) \otimes \mathbb{R} \) is the direct sum of all the groups \( H^p(X; \mathbb{R}) \) for \( p \) even, where \( H^p(X; \mathbb{R}) \) denotes the \( p \)th Čech cohomology group of \( X \) with real coefficients. (This is a consequence of [2, p. 19] and the universal coefficient theorem. A more
elementary proof is contained in [1, §3.2]; here, however, we have to invoke the Eilenberg-Steenrod uniqueness theorem to deduce that the \( H^p \)'s are in fact Čech cohomology.) It now follows immediately from Proposition 2 that:

**Corollary 3.** For any compact space \( X \) there is a natural isomorphism

\[
K_M(X) \cong \bigoplus \{ H^p(X; \mathbb{R}) : p \text{ even}, p > 0 \}.
\]

2. Let \( X \) be a compact metric space. An extension of \( C(X) \) relative to \( M \) is a unital *-monomorphism \( \tau : C(X) \to \mathbb{A}(M) \). Two such extensions \( \tau_1, \tau_2 \) are equivalent if there is an inner automorphism \( \alpha \) of \( M \) (which maps \( \mathbb{K}(M) \) onto \( \mathbb{K}(M) \) and so induces an automorphism \( \tilde{\alpha} \) of \( \mathbb{A}(M) \)) with \( \tau_2 = \tilde{\alpha} \circ \tau_1 \). The set \( \text{Ext}^M(X) \) of equivalence classes of extensions of \( C(X) \) relative to \( M \) is a group (see [12]), is a homotopy invariant functor of the space \( X \) and can be used to define a generalised homology theory (see [9]). Cho also proves in [9] that \( \text{Ext}^M \) is naturally equivalent to \( \text{Hom}(\tilde{K}(\cdot)), \mathbb{R}) \)–and so is independent of \( M \).

**Proposition 4.** For any compact metric space \( X \) there is a natural isomorphism

\[
\text{Ext}^M(X) \cong \prod \{ H_p(X; \mathbb{R}) : p \text{ odd}, p > 1 \}
\]

where \( H_p(X, \mathbb{R}) \) denotes the \( p \)th Čech homology group of \( X \) with real coefficients.

**Proof.** First we suppose that \( X \) is a compact polyhedron. By the main theorem of [9], \( \text{Ext}^M(X) \cong \text{Hom}(\tilde{K}(SX); \mathbb{R}) \) where \( SX \) denotes the unreduced suspension of \( X \). This in turn can be identified with \( \text{Hom}_R(\tilde{K}(SX) \otimes \mathbb{R}, \mathbb{R}) \), which is isomorphic to \( \text{Hom}_R(\bigoplus_{p \text{ even}} \tilde{H}^p(SX; \mathbb{R}), \mathbb{R}) \) by Corollary 3. Since \( X \) is a compact polyhedron \( \text{Hom}(\tilde{H}^p(SX); \mathbb{R}) \cong H_p(SX) \) [13, 23.14] and so \( \text{Ext}^M(X) \cong \prod_{p \text{ even}} \tilde{H}_p(SX; \mathbb{R}) \). But \( \tilde{H}_p(SX) \cong H_{p-1}(X) \), and we have the result for compact polyhedra. The general case now follows by observing that both \( \text{Ext}^M \) and \( \prod H_p(\cdot, \mathbb{R}) \) are continuous functors [9, Corollary 1].

**Remarks.** Although the Čech homology theory \( H_\ast(\cdot, G) \) with coefficients in an abelian group \( G \) satisfies the continuity axiom, it does not in general have a long exact sequence; the appropriate theory for compact metric spaces is Steenrod homology, denoted \( ^sH_\ast(\cdot, G) \). In addition to the seven Eilenberg-Steenrod axioms, \( ^sH_\ast \) satisfies the relative homeomorphism axiom and the cluster axiom (see [15] or [16]), and is characterised uniquely by these axioms [16, Theorem 3]. For an arbitrary coefficient group Čech homology satisfies all these axioms except exactness; however, when the coefficient group is \( \mathbb{R} \), Čech homology is exact [11, Theorem IX.7.6] and so coincides with Steenrod homology on compact metric spaces. Thus the last proposition is valid with the Čech groups \( H_\ast(X; \mathbb{R}) \) replaced by the corresponding Steenrod groups.
For further details on the relationships between Čech and Steenrod homology we refer to [15] and [16]; the Čech theory is discussed in detail in [11].

In [9] Cho proves the Ext^f is a generalised Steenrod theory which is also continuous; hence Ext^f is also a generalised Čech theory. (Here by a generalised theory we mean one which satisfies all the appropriate axioms except dimension; the axioms for Čech homology are given in [11, Chapter X].) This is not the case for the Brown-Douglas-Fillmore theory Ext^f; it is a generalised Steenrod theory but is not continuous—in fact Brown [7] has shown that it fails to be continuous in the same way as the Steenrod homology theory. This is discussed in [15].

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REFERENCES


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