LOCAL $p$-SIDON SETS FOR LIE GROUPS

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ABSTRACT. It is shown that a compact Lie group admits no local $p$-Sidon sets of unbounded degree.

Let $G$ be a compact group, and let $1 < p < 2$. A subset $R$ of the dual of $G$ is called a local $p$-Sidon set if there exists a constant $B$ such that for every $\sigma \in R$ and for every $d_0 \times d_0$ matrix $A_\sigma$,

$$\|A_\sigma\|_p \leq Bd_0^{1/p}\|\text{tr} A_\sigma(\cdot)\|_\infty.$$  (1)

THEOREM. If $G$ is a compact Lie group, and if $R$ is a local $p$-Sidon set for $G$, then $\sup\{d_0|\sigma \in R\} < \infty$.

PROOF. We first note that, if $G$ is an arbitrary compact group, $R$ is a $p$-Sidon set for $G$, and if $r > 1$, then there exists a constant $\kappa_r$ such that for all $\sigma \in R$

$$\|\chi_\sigma\|_r \leq \kappa_r d_0^{2/r}$$  (2)

where $\chi_\sigma(x) = \text{tr}(\sigma(x))$.

To see this, we first use a simple duality argument to see that (1) is equivalent to: there exists a constant $C$ such that for every $\sigma \in R$ and for every $d_0 \times d_0$ matrix $A_\sigma$, there exists $g \in L^1(G)$ such that $\hat{g}(\sigma) = A_\sigma$, and $\|g\|_1 \leq Cd_0^{1/p}\|A_\sigma\|_p$. Thus for every $\sigma \in R$ and for every $d_0 \times d_0$ unitary matrix $W$, there exists $g_W \in L^1(G)$ with $\hat{g}_W = W^*$, and $\|g_W\|_1 \leq Cd_0^{1/p}\|W^*\|_p = d_0^{2/p}$. Since $\chi_\sigma = g_W \ast (\text{tr}(W \cdot \sigma(x)))$ we have

$$\|\chi_\sigma\|_r \leq \|g_W\|_1 \left(\int_G |\text{tr}(W \cdot \sigma(x))| \, dx\right)^{1/r}$$

$$\leq Cd_0^{2/p}\left(\int_G |\text{tr}(W \cdot \sigma(x))| \, dx\right)^{1/r}.$$  (3)

Hence, integrating over the $d_0 \times d_0$ unitary group, $\mathcal{U}(d_0)$ with respect to normalized Haar measure $dW$, and using Hölder's inequality, we obtain

$$\|\chi_\sigma\|_r \leq Cd_0^{2/p}\left(\int_G \int_{\mathcal{U}(d_0)} |\text{tr}(W \cdot \sigma(x))| \, dW \, dx\right)^{1/r}$$

$$= Cd_0^{2/p}\left(\int_{\mathcal{U}(d_0)} |\text{tr} W| \, dW\right)^{1/r}.$$
The last equality follows from the translation invariance of $dW$. It is easily established, however (cf. [2, (29.12)]), that there exists a bound $K_r$, independent of $d$, for $(\int |W|^r dW)^{1/r}$. Thus we obtain (2).

Suppose now that $G$ is a compact Lie group. In [1, Theorem (5.4)], the following estimate is given for the $r$-norms of the irreducible characters; let $M_G \in \mathbb{R}$ be as in [1, (5.5)]. Then for $r > M_G$, there exists a constant $\kappa$, such that

$$\kappa d^{1-M_G/r}_\sigma \leq \|\chi_\sigma\|_r. \quad (3)$$

From (2) and (3), it follows that, for all $r > M_G$, $\sup_{\sigma \in \pi} d^{1-2/p'-M_G/r}_\sigma < \infty$, and hence, since $p < 2$, $\sup_{\sigma \in \pi} d_\sigma < \infty$. □

It follows that, if $G$ is a compact semisimple Lie group, $G$ has no infinite local $p$-Sidon sets.

It should be noted that a set $R$ with $\sup \{d_\sigma | \sigma \in R\} < \infty$ is local Sidon and hence local $p$-Sidon for all $p$ [3].

REFERENCES

4. P. M. Soardi, $\mathcal{S}_{\mathbb{R}_2}$ has no infinite local $p$-Sidon sets (preprint).

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