AN ERGODIC THEOREM FOR
FRÉCHET-VALUED RANDOM VARIABLES

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Abstract. We generalize the classical ergodic theorem from real-valued random variables to Fréchet space-valued random variables and obtain this generalization as a direct corollary of the classical theorem. As an application we obtain several strong laws of large numbers for Fréchet-valued random variables. In a similar way we obtain a martingale theorem for Fréchet-valued random variables.

1. Preliminaries. Let $F$ be a Fréchet space and $q_k$, $k \in \mathbb{N}$, be a family of seminorms generating the topology of $F$. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

We shall consider the concept of integration according to Schäfke [4], [5], [6]. Let $\mathcal{S} = \{ \sum_{r=1}^{n} x_r I_{A_r} : x_r \in F, A_r \in \mathcal{F} \}$ be the system of $F$-valued simple functions. The integral of a simple function $X = \sum_{r=1}^{n} x_r I_{A_r}$ is defined by $E(X) = \sum_{r=1}^{n} P(A_r) x_r$.

We can define the integral norm $\| \| : [0, \infty] \rightarrow [0, \infty]$ (in Schäfke's notation $\| \|_3$) according to Theorem 5.6.1 of [5]. In our special case of a probability measure this integral norm is given by

$$\|g\| = \inf \{ E(h) : g < h, h \text{ \(\mathcal{F}\)-measurable} \},$$

where $E(h)$ is the classical integral of an $\mathcal{F}$-measurable function $h$. By

$$\rho^*(X, Y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|q_k(X - Y)\|}{1 + \|q_k(X - Y)\|}$$

there is defined a pseudometric in the space of all functions from $\Omega$ to $F$. Then a function $X : \Omega \rightarrow F$ is $P$-integrable if there exists a sequence of simple functions $X_n : \Omega \rightarrow F$ such that $\rho^*(X_n, X) \rightarrow_{n \rightarrow \infty} 0$. Then $E(X_n)$, $n \in \mathbb{N}$, is Cauchy-convergent in $F$ and $E(X) = \lim_{n \rightarrow \infty} E(X_n)$ is the $P$-integral of an $P$-integrable function $X$.

Let $L_1(\Omega, \mathcal{F}, P, F)$ be the system of all $F$-valued $P$-integrable functions. According to Theorem 2.4.5 of [5] we have $X \in L_1(\Omega, \mathcal{F}, P, F)$ iff there exists a sequence $X_n \in \mathcal{F}$, $n \in \mathbb{N}$ such that

1. $X_n \rightarrow X$ $P$-a.e.
2. $E(q_k(X_n - X_m)) \rightarrow_{n,m \rightarrow \infty} 0$ for all $k$.

Hence the range of an $P$-integrable function is contained $P$-a.e. in a separable Fréchet space. Let $\mathcal{F}_0$ be a sub-$\sigma$-field of $\mathcal{F}$ and $X \in L_1(\Omega, \mathcal{F}_0, P, F)$. Using

Received by the editors July 6, 1977 and, in revised form, December 12, 1977.

AMS (MOS) subject classifications (1970). Primary 28A65; Secondary 60F15.

Key words and phrases. Pointwise ergodic theorem, strong law of large numbers, weakly orthogonal processes.
the same techniques as Neveu [9, p. 100] for functions with values in a separable Banach space, one can define the conditional expectation \( E(X|\mathcal{A}_0) \).

2. The results. We show that the classical ergodic theorem holds true also for Fréchet-valued integrable functions. For the proof of this theorem we only use the classical ergodic theory for real-valued functions. We can immediately derive some strong laws of large numbers for Banach-valued and Fréchet-valued random variables which have been proved before by heavy techniques. With the same technique we show that a martingale theorem for Fréchet-valued random variables holds true. Compare with the heavy techniques used in [1, Satz 22.2] for Banach-valued variables.

**Theorem 1.** Let \( T \) be a measure preserving transformation of the probability space \((\Omega, \mathcal{F}, P)\). Let \( F \) be a Fréchet space and \( X: \Omega \to F \) be a \( P \)-integrable function. Then

\[
\frac{1}{n} \sum_{k=0}^{n-1} X \circ T^k \to E(X|\mathcal{G}_T) \quad P\text{-a.e.}
\]

where \( \mathcal{G}_T = \{ A \in \mathcal{F} : T^{-1}(A) = A \} \) is the \( \sigma \)-algebra of \( T \)-invariant sets.

**Proof.** According to the classical ergodic theorem the assertion is true for all characteristic functions \( X = 1_A \) with \( A \in \mathcal{F} \) and hence for the system \( \mathcal{F} \) of all simple functions. Now let \( X \in L^1(\Omega, \mathcal{F}, P, F) \). Then there exist \( X_j \in \mathcal{F}, j \in \mathbb{N} \), such that

\[
\rho^*(X_j, X) \to 0 \quad j \to \infty
\]

and

\[
E(X_j|\mathcal{G}_T) \to E(X|\mathcal{G}_T) \quad P\text{-a.e.}
\]

We have for all \( j, k \in \mathbb{N} \):

\[
q_k \left( \frac{1}{n} \sum_{r=0}^{n-1} X(T^r(\omega)) - E(X|\mathcal{G}_T) \right)
\]

\[
< q_k \left( \frac{1}{n} \sum_{r=0}^{n-1} (X - X_j)(T^r(\omega)) \right) + q_k \left( \frac{1}{n} \sum_{r=0}^{n-1} X_j(T^r(\omega)) - E(X|\mathcal{G}_T) \right) + q_k (E(X_j|\mathcal{G}_T) - E(X|\mathcal{G}_T)).
\]

Since the assertion of the theorem holds for all \( X_j \) we obtain for all \( j, k \in \mathbb{N} \):

\[
\lim_{n \to \infty} q_k \left( \frac{1}{n} \sum_{r=0}^{n-1} X(T^r(\omega)) - E(X|\mathcal{G}_T) \right)
\]

\[
< \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} q_k((X - X_j)(T^r(\omega))) + q_k (E(X_j|\mathcal{G}_T) - E(X|\mathcal{G}_T)). \quad (3)
\]

Let
\[ f_j^{(k)}(\omega) = q_k((X - X_j)(\omega)) \]

and

\[ g_j^{(k)}(\omega) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} f_j^{(k)}(T^r(\omega)). \]

By the classical ergodic theorem

\[ g_j^{(k)} = E\left(f_j^{(k)}|\mathcal{F}_T\right) \quad \text{P-a.e.} \]

and

\[ E\left(E\left(f_j^{(k)}|\mathcal{F}_T\right)\right) = E\left(f_j^{(k)}\right) \to 0 \]

as \( j \to \infty \) by (1). Therefore

\[ g_j^{(k)} = E\left(f_j^{(k)}|\mathcal{F}_T\right) \to 0 \quad \text{P-a.e.} \]

for a subsequence \( j \in \mathbb{N}_0. \) Now the assertion follows from (2) and (3).

**Theorem 2.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(\mathcal{F}_n, n \in \mathbb{N},\) be a sequence of sub-\(\sigma\)-fields of \(\mathcal{F}\) decreasing or increasing to the \(\sigma\)-field \(\mathcal{F}_\infty.\) Let \(F\) be a Fréchet space and \(X: \Omega \to F\) be a \(\mathbb{P}\)-integrable function. Then

\[ E(X|\mathcal{F}_n) \to E(X|\mathcal{F}_\infty) \quad \text{P-a.e.} \]

**Proof.** According to the classical martingale theorem the theorem is true for all characteristic functions \(X = 1_A\) with \(A \in \mathcal{F}\) and hence for the system \(\mathcal{F}\) of all simple functions. Now let \(X \in L_1(\Omega, \mathcal{F}, \mathbb{P}, F)\). Then there exist \(X_j \in \mathcal{F}, j \in \mathbb{N},\) such that

\[ \rho^*(X_j, X) \to 0. \quad (1) \]

We have for all \(j, k, n \in \mathbb{N} \)

\[ q_k(E(X|\mathcal{F}_n) - E(X|\mathcal{F}_\infty)) \]

\[ \leq q_k(E(X - X_j|\mathcal{F}_n)) + q_k(E(X_j|\mathcal{F}_n) - E(X_j|\mathcal{F}_\infty)) \]

\[ + q_k(E(X_j - X|\mathcal{F}_\infty)) \]

\[ \leq E(q_k(X - X_j)|\mathcal{F}_n) + q_k(E(X_j|\mathcal{F}_n) - E(X_j|\mathcal{F}_\infty)) \]

\[ - E(X_j|\mathcal{F}_\infty)) + E(q_k(X_j - X)|\mathcal{F}_\infty)). \]

Since the assertion of the theorem holds for all \(X_j,\) we obtain for all \(j, k \in \mathbb{N}:\)

\[ \lim_{n \to \infty} q_k(E(X|\mathcal{F}_n) - E(X|\mathcal{F}_\infty)) \]

\[ \leq \lim_{n \to \infty} E(q_k(X - X_j)|\mathcal{F}_n) + E(q_k(X_j - X)|\mathcal{F}_\infty). \]

Using the classical martingale theorem we obtain for all \(j, k \in \mathbb{N}:\)

\[ \lim_{n \to \infty} q_k(E(X|\mathcal{F}_n) - E(X|\mathcal{F}_\infty)) \leq 2E(q_k(X_j - X)|\mathcal{F}_\infty). \quad (2) \]

By (1) for all \(k \in \mathbb{N}:\)
for a subsequence \( j \in \mathbb{N} \). Hence the assertion follows from (2).

It is also possible to obtain in this direct way convergence results for convergence in the \( p \)-th mean for Theorems 1 and 2.

Considering the canonical process associated to a stochastic process one obtains from Theorem 1 as in the classical case:

**Corollary 3.** Let \( X_n, n \in \mathbb{N} \), be a stationary process defined on a probability space \( (\Omega, \mathcal{F}, P) \) with values in a Fréchet space \( F \). Let \( X_1 \) be \( P \)-integrable, then

\[
\frac{1}{n} \sum_{k=1}^{n} X_k \to_{n \to \infty} E(X_1 | \mathcal{F}(X_n: n \in \mathbb{N})) \quad P\text{-a.e.}
\]

where \( \mathcal{F}(X_n: n \in \mathbb{N}) \) is the system of all invariant sets of the process \( X_n, n \in \mathbb{N} \).

Since \( \mathcal{F}(X_n: n \in \mathbb{N}) \) is contained in the \( \sigma \)-field of terminal sets of the process \( X_n, n \in \mathbb{N} \), the zero-one law of Kolmogorov implies that every invariant set has measure 0 or 1, if \( X_n, n \in \mathbb{N} \), are independent. Hence we obtain

**Corollary 4.** Let \( X_n, n \in \mathbb{N} \), be independent and identically distributed random variables with values in a Fréchet space \( F \). Let \( X_1 \) be \( P \)-integrable then

\[
\frac{1}{n} \sum_{k=1}^{n} X_k \to_{n \to \infty} E(X_1) \quad P\text{-a.e.}
\]

Using the fact that a measurable random variable \( X \), with values in a separable Fréchet space, fulfilling \( E[q_k(X)] < \infty \) for each \( k \in \mathbb{N} \), belongs to \( L_1(\Omega, \mathcal{F}, P, F) \), we obtain the theorem of Taylor and Padgett [7]. Hence we obtain Theorems 4.1.1 and 6.1.2 of [3] as special cases. We can also generalize Theorem 4.3.1 of [3] from Banach spaces with a separable dual to Fréchet spaces with a separable dual.

**Corollary 5.** Let \( X_n, n \in \mathbb{N} \), be a stationary process defined on a probability space with values in a Fréchet space with a separable dual space. Let \( E(X_1) = 0 \), \( E(q_k(X_1)) < \infty \) for all \( k \in \mathbb{N} \). Let the process be weakly orthogonal, i.e. \( E(f(X_n)f(X_m)) = 0 \) for all \( n \neq m \) and all continuous linear functionals of \( F \), then

\[
\frac{1}{n} \sum_{k=1}^{n} X_k \to 0 \quad P\text{-a.e.}
\]

**Proof.** According to Corollary 2 we obtain

\[
\frac{1}{n} \sum_{k=1}^{n} X_k \to_{n \to \infty} E(X_1 | \mathcal{F}(X_n: n \in \mathbb{N})).
\]
Now using the same method as in [3, p. 56] we obtain the assertion.

REFERENCES


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