THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS
WITH VARIABLE MULTIPLE CHARACTERISTICS

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Abstract. Let \( P(t, x, D_t, D_x) \) be a hyperbolic differential operator with the principal symbol \( p_m(t, x, \tau, \xi) \). We assume that \( p_m \) is denoted by \( \prod_{j=1}^{s} (\tau - \lambda_j)^{m_j} \prod_{j=s+1}^{m} (\tau - \lambda_j) \) and \((\lambda_j(t, x, \xi) \neq 0 \text{ if } (i, j) \neq (k, m - N + k) (k = 1, \ldots, s)\), where \( N = \sum_{j=1}^{s} m_j \) and \( \lambda_j(t, x, \xi) \in C^\infty([0, T] \times R^n \times (R^n \setminus 0)) \). Under a generalized condition of E. E. Levi, we shall show that the Cauchy problem \( Pu = f \) in \([0, T] \times R^n, \psi = \psi \) \( (\xi = 1, \ldots, m - 1) \) is well posed. When \( m_j = 1 (j = 1, \ldots, s) \), our result coincides those of Ohya and Petkov.

1. Statement of the result. We shall consider the following differential operator:

\[
P(t, x, D_t, D_x) = \sum_{|\alpha| < m} a_\alpha(t, x)D_t^{a_\alpha}D_x^{a_\alpha},
\]

where \((t, x) \in [0, T] \times R^n (0 < T < \infty), D_t^{a_\alpha}D_x^{a_\alpha} = D_t^{a_\alpha_1} \cdots D_x^{a_\alpha_n}, D_t = -i\partial/\partial t, D_x = -i\partial/\partial x_j\). We assume that \( a_{m_0}, \ldots, a_{m_0}(t, x) = 1 \) and \( a_\alpha(t, x) \) belongs to \( \mathcal{S}([0, T] \times R^n) \), which consists of all functions having that these arbitrary derivations are bounded in \([0, T] \times R^n\). Let \((\tau, \xi)\) be the covariable of \((t, x)\). Then we define the following symbols:

\[
p_k(t, x, \tau, \xi) = \sum_{|\alpha| = k} a_\alpha(t, x)\tau^{a_\alpha}\xi^{a_\alpha} \quad (k = 0, \ldots, m).
\]

First we shall impose the following assumptions on \( p_m \):

\[
p_m(t, x, \tau, \xi) = \prod_{j=1}^{s} (\tau - \lambda_j)^{m_j} \prod_{j=s+1}^{m} (\tau - \lambda_j),
\]

where \( m_1 > m_2 > \cdots > m_s, N = \sum_{j=1}^{s} m_j \) and all functions \( \lambda_j(t, x, \xi) (j = 1, \ldots, m - N + s) \) are real and positively homogeneous of degree 1 with respect to \( \xi \) and belong to \( \mathcal{B}([0, T] \times R^n \times S^{n-1}) \) if \( \xi \in S^{n-1} \). Here \( S^{n-1} \) is the unit sphere of \( R^n \).

(A.2) For any couple \((i, j) \neq (k, m - N + k) (k = 1, \ldots, s)\) we suppose the following:

\[
|\lambda_i - \lambda_j|(t, x, \xi) > C, \quad (t, x, \xi) \in [0, T] \times R^n \times S^{n-1},
\]

where \( C \) is a positive constant.

Throughout this note, the symbols of pseudo-differential operators are elements of \( \mathcal{B}([0, T] \times R^n \times S^n) \) or \( \mathcal{B}([0, T] \times R^n \times S^{n-1}) \) if \((\tau, \xi) \in S^n \) or
Moreover we use the following notation. For symbols $a(t, x, \xi)$, $b(t, x, \xi)$ and $A(t, x, \tau, \xi)$, $A_{\tau=a} \equiv 0 \mod b$ means that there exists a symbol $c(t, x, \xi)$ such that

$$A(t, x, a(t, x, \xi), \xi) = c(t, x, \xi)b(t, x, \xi).$$

For the lower order terms of $P$ we assume the following:

(A.3) For any $k$ ($k = 1, \ldots, s$) we can denote $P(t, x, D_t, D_x)$ by the following form:

$$P = \sum_{l=0}^{m_k} Q_{k,l}(t, x, D_t, D_x)(\Lambda_k(t, x, D_t, D_x))^{l}. \quad (1.1)$$

Here $\Lambda_k$ is the pseudo-differential operator defined by the symbol $\tau - \lambda_k(t, x, \xi)$ and $Q_{k,l}$ ($l = 0, \ldots, m_k$) is a pseudo-differential operator of order $m - m_k$, whose principal symbol $q_{k,l}(t, x, \tau, \xi)$ has the following property:

$$q_{k,l}(\tau - \lambda_k) \equiv 0 \mod \lambda_k - \lambda_{m - N + k}. \quad (1.2)$$

Clearly if $\lambda_k = \lambda_{m - N + k}$, then the above condition (A.3) is that of E. E. Levi. In the final part of this note we denote (A.3) by the condition with respect to $p_k$, when $m_k = 1$ or 2.

For a nonnegative integer $k$ and $s \in \mathbb{R}$ the function space $C^k([0, T]; H_s(R^n))$ consists of functions such that $D^k u(t) (j = 0, \ldots, k)$ exists as an element of $H_s_j(R^n)$ and is continuous on the topology of $H_s_j(R^n)$. We use the following norm:

$$|||u(t)|||_{s,k}^2 = \sum_{j=0}^{k} ||D^j u(t)||_{s-j}^2$$

where $|| \cdot ||_{s-j}$ is the usual norm of $H_{s-j}(R^n)$.

Now we can state our theorem.

**Theorem.** Let $P(t, x, D_t, D_x)$ be a differential operator of order $m$. If $P$ satisfies the assumptions (A.1), (A.2) and (A.3), then the Cauchy problem $Pu = f$ in $[0, T] \times R^n$, $D^m u|_{t=0} = g_j (j = 0, \ldots, m - 1)$ is well posed, i.e., for $f \in C^{k-m+m_1+1}(0, T)$; $H_{s-m+m_1+1}(R^n)$ and $g_j \in H_{s-j+m_1}(R^n)$ there exists a unique solution $u(t, x) \in C^k([0, T]; H_s(R^n))$ such that

$$|||u(t)|||_{s,k} \leq C \left\{ \sum_{j=0}^{m-1} ||g_j||_{s-j+m_1} + ||f(0)||_{s-m+m_1-k-m+m_1} + \int_0^t ||f(r)||_{s-m+m_1+1,k-m+m_1+1} dr \right\},$$

where $k > m - m_1 - 1$, $||f(0)||_{s-m+m_1-1} = 0$ and $t \in [0, T]$.

This Theorem is the same as those of [3] and [4] when $m_j = 1$ ($j = 1, \ldots, s$). Under a different situation, in [1] they consider the Cauchy problem of a triple case.
2. Reform of the condition (A.3). In this section, we state an equivalent condition of (A.3).

Taking care of multiplicities of the roots \( \lambda_j \) \((j = 1, \ldots, s)\), we can denote \( p_m(t, x, \tau, \xi) \) by

\[
(\tau - \lambda_{m-N+s}) \cdots (\tau - \lambda_{s+1}) \Phi_{\mu} \cdots \Phi_1, 
\]

where \( \Phi_{\nu}(t, x, \tau, \xi) \) \((\nu = 1, \ldots, \mu)\) is a polynomial of degree \( s \) with respect to \( \tau \) and equal to \( \Pi_{j=1}^s (\tau - \lambda_j) \). Here \( m_{s-1}+1 = \cdots = m_\mu \) \((\nu = 1, \ldots, \mu)\).

Remark that \( s_1 < s_2 < \cdots < s_\mu = s \), \( m_1 = \Sigma_{\nu=1}^s m_\nu \) and denote \( N_\nu \) by \( s_\nu m_\nu \).

We introduce a product pseudo-differential operator \( \Phi_{\nu}(t, x, D_t, D_x) = (\Lambda_s \cdots \Lambda_1)(t, x, D_t, D_x) \). Then we denote \( \Delta_j(t, x, D_t, D_x) \) of order \( j \) \((j = 0, \ldots, m)\) by \( \Delta_0 = 1, \Delta_1 = \Lambda_1, \ldots, \Delta_j, \ldots, \Delta_N = \Phi_{\mu} \cdots \Phi_1, \ldots, \Delta_{m+j} \).

Proposition 2.1. Let \( P(t, x, D_t, D_x) \) be a differential operator which satisfies the conditions (A.1) and (A.2). Then the condition (A.3) is equivalent to the following statement. We can denote \( P \) by

\[
P(t, x, D_t, D_x) = \sum_{i=0}^{m_1} Q_i(t, x, D_t, D_x) \Delta_I(i), 
\]

where if \( i = n_\mu + \cdots + n_{s+1} + \sigma \) \((1 \leq \sigma \leq n_s)\), then \( I(i) = N_1 + \cdots + N_\nu - \sigma \sigma \) and \( Q_i \) \((i = 0, \ldots, m_1)\) is a pseudo-differential operator of order \( M_i = m - i - I(i) \) and differential operator of \( t \). Moreover the principal symbol \( q_I(t, x, \tau, \xi) \) of \( Q_i \) satisfies the following condition:

\[
q_{\nu\nu - \lambda_\nu} \equiv 0 \mod \lambda_k - \lambda_{m-N+k} \quad \text{if} \quad k < s_\nu. 
\]

Since any pseudo-differential operator of order \( l \) \((l < m)\) which is a differential operator of \( t \) is represented by \( \Delta_0, \ldots, \Delta_m \), we have the following:

Proposition 2.2. Let \( P \) be a differential operator which satisfies the conditions (A.1), (A.2), (2.1) and (2.2). Then \( P \) is denoted by

\[
P = \sum_{i=0}^{m_1} \sum_{j=0}^{M_i} R_{i,j}(t, x, D_x) \Delta_I(i) + j, 
\]

where \( R_{i,j} \) is a pseudo-differential operator of order \( M_i - j \) whose principal symbol is \( r_{i,j}(t, x, \xi) \), and \( r_{i,j} \) have the following property:

\[
\sum_{j=0}^{k-1} r_{i,j} \Lambda_j \cdots \Lambda_{1|\nu - \lambda_\nu} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \quad k < s_\nu, 
\]

if \( i = n_\mu + \cdots + n_{s+1} + \sigma \) \((1 \leq \sigma \leq n_\nu)\).

3. Reduction to a first order system and the proof of the Theorem. Since the proof of the Theorem is inferred on the analogy of a simple case, we assume that \( s = 2, m_1 = 2 \) and \( m_2 = 1 \). Thus by (2.3) our considered operator
$P(t, x, D_t, D_x)$ in (1.1) is denoted by

$$P(t, x, D_t, D_x) = \Delta_m + \sum_{i=1}^{2} \sum_{j=0}^{m-i} R_j^{m-i} \Delta_n$$

where $R_j^{m-i}$ is of order $m - i - j$. The condition (2.4) says that

$$R_j^{m-i}(t, x, D_t) = 0, \quad (3.1)$$

where $R_j^{m-i}$ is of order $m - i - j$. The condition (2.4) says that

$$R_j^{m-i}(t, x, D_t) = 0, \quad (3.2)$$

and

$$(r_1^{m-1} + r_2^{m-1}(\lambda_2 - \lambda_1))(t, x, \xi) \equiv 0 \mod \lambda_2 - \lambda_{m-1}. \quad (3.3)$$

We denote a pseudo-differential operator with the symbol $|\xi|$ by the term $\Lambda$ and define a column vector

$$U = \left(\Lambda^{m-3} \Delta_0 u, \Lambda^{m-3} \Delta_1 u, \Lambda^{m-4} \Delta_2 u, \ldots, \Lambda \Delta_{m-2} u, \Delta_{m-1} u\right) \quad (3.4)$$

and $F = \left(0, \ldots, 0, f\right)$. Then by (3.1) the equation $Pu = f$ becomes the following first order system:

$$MU = (\partial_t - A(t, x, D_x))U + B(t, x, D_x)U = F,$$

where $B$ is of order 0 and $A$ is a first order pseudo-differential operator with the symbol

$$\alpha(t, x, \xi) = \begin{pmatrix}
\lambda_1, 0, & 0, & 0,
\lambda_1, |\xi|, & 0, & 0,
\lambda_2, 0, & 0, & 0,
\lambda_3, |\xi|, & 0, & 0,
a_{m1}, a_{m2}, a_{m3}, 0, 0, 0, 0, & 0, & 0, & \lambda_{m-1}
\end{pmatrix},$$

where $a_{m1} = -r_0^{m-2}(t, x, \xi/|\xi|)|\xi|$, $a_{m2} = -r_1^{m-1}(t, x, \xi/|\xi|)|\xi|$, and $a_{m3} = -r_2^{m-1}(t, x, \xi/|\xi|)|\xi|$. Then $\alpha$ has the following property.

**Proposition 3.1.** There exists a nonsingular matrix $N(t, x, \xi)$ whose components are homogeneous of degree 0 and belong to $\mathcal{B}([0, T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$ such that

$$(aN)(t, x, \xi) = (ND)(t, x, \xi),$$

where $D = (d_{ij})$ is a $m \times m$ diagonal matrix with $d_{11} = d_{22} = \lambda_1$ and $d_{ii} = \lambda_{i-1}$ ($i = 2, \ldots, m$).
Proof. Since clearly an eigenvalue of \( a(t, x, \xi/|\xi|) \) is \( \lambda_j = \lambda_j(t, x, \xi/|\xi|) \) \((j = 1, \ldots, m - 1)\), we shall seek an eigenvector of \( \lambda_j \). The equation

\[
(\mu I - a(t, x, \xi/|\xi|))n = 0,
\]

where \( n = (n_1, \ldots, n_m) \), is equivalent to the following:

\[
\begin{align*}
(\mu - \bar{\lambda}_1)n_1 &= 0, \\
(\mu - \bar{\lambda}_1)n_2 - n_3 &= 0, \\
(\mu - \bar{\lambda}_2)n_3 &= 0,
\end{align*}
\]

\[
(\mu - \bar{\lambda}_{m-1})n_m + \bar{r}_0^{m-2}n_1 + \bar{r}_1^{m-1}n_2 + \bar{r}_2^{m-1}n_3 = 0,
\]

(3.5)

where \( \bar{r}_j^{m-i}(t, x, \xi) = r_j^{m-i}(t, x, \xi/|\xi|) \). Since \( \bar{r}_1^{m-1}n_2 + \bar{r}_2^{m-1}n_3 = (\bar{r}_1^{m-1} + \bar{r}_2^{m-1}(\mu - \lambda_1))n_2 \), for example in fact we can take the following matrix \( N(t, x, \xi) \) such that each column of \( N \) is an eigenvector of an eigenvalue \( \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{m-1} \) respectively in turn;

\[
N(t, x, \xi) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & (\lambda_2 - \lambda_1)^{-1} & 0 \\
0 & 0 & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
1 & 0 & \cdots & \cdots \\
* & * & \cdots & \cdots \\
n_{m-11}, n_{m-12}, n_{m-13} & 0 & \cdots & 1 \\
n_{m1}, n_{m2}, n_{m3} & 0 & \cdots & 1
\end{pmatrix}
\]

where

\[
\begin{align*}
n_{m1} &= \bar{r}_0^{m-2}/(\bar{\lambda}_1 - \bar{\lambda}_{m-1}), \\
n_{m-11} &= \bar{r}_0^{m-2}/(\bar{\lambda}_1 - \bar{\lambda}_{m-1})(\bar{\lambda}_1 - \bar{\lambda}_{m-2}), \\
n_{m2} &= \bar{r}_1^{m-1}/(\bar{\lambda}_1 - \bar{\lambda}_{m-1}), \\
n_{m-12} &= \bar{r}_1^{m-1}/(\bar{\lambda}_1 - \bar{\lambda}_{m-1})(\bar{\lambda}_1 - \bar{\lambda}_{m-2}), \\
n_{m3} &= (\bar{r}_1^{m-1} + \bar{r}_2^{m-1}(\bar{\lambda}_2 - \bar{\lambda}_1))/(\bar{\lambda}_2 - \bar{\lambda}_{m-1}), \\
n_{m-13} &= n_{m3}/(\bar{\lambda}_2 - \bar{\lambda}_{m-2}).
\end{align*}
\]

These are well defined by (A.2), (3.2) and (3.3) and \( * \) are easily inductively determined from (A.2) and (3.5). Clearly the matrix \( N(t, x, \xi) \) is positively homogeneous of degree 0 and a nonsingular matrix in \([0, T] \times R^n \times (R^n \setminus 0)\). This completes the proof of Proposition 3.1.

By Proposition 3.1 the following fact is well known (see [2]), which guarantees the validity of our theorem.

Proposition 3.2. For any nonnegative integer \( k \) and \( s \in R \) if \( F(t, x) \in C^k([0, T]; H_s(R^n)) \) and \( U_0(x) \in H_s(R^n) \), then there exists a unique solution \( U(t, x) \in C^k([0, T]; H_s(R^n)) \) of the Cauchy problem for \( M \) with initial data on \( t = 0 \) such that

\[
\|U(t)\|_{s,k} < C\left(\|U(0)\|_{s,k} + \int_0^T\|MU(r)\|_{s,k}dr\right),
\]
where \( C \) does not depend on \( U(t,x) \) and \( t \in [0, T] \).

**Remark.** For a general case the vector \( U \) of (3.4) becomes the following:
\[
U = (\Lambda^{m-1} \Delta_{0} u, \ldots, \Lambda^{n-1} \Delta_{n}, \ldots, \Lambda^{m-N-1} \Delta_{n}, \ldots, \Lambda \Delta_{m-N-1} \Delta_{m-1}),
\]
where \( a_{j} = m - (n_{j} + \cdots + n_{p}) + \sigma - j - 1 \) if \( j = N_{1} + \cdots + N_{p-1} + s_{i} \sigma + \delta \) (\( \sigma = 0, \ldots, n_{j} - 1, \delta = 0, \ldots, s_{i} - 1 \)).

4. Some sufficient condition of (A.3) when \( m_{k} = 1 \) or 2. In this section we shall denote (A.3) by the conditions with respect to \( p_{k} \) when \( m_{k} = 1 \) or 2. We denote the subprincipal symbol of \( P \) by
\[
P_{m-1}(x, \xi) = p_{m-1} + (i/2) \sum_{j=0}^{n} \frac{\partial \tilde{f}_{m}}{\partial x_{j} \partial \xi_{j}}
\]
and the Poisson bracket of \( f(x, \xi) \) and \( g(x, \xi) \) by
\[
\{f, g\}(x, \xi) = \sum_{j=0}^{n} \left( \frac{\partial f}{\partial \xi_{j}} \frac{\partial g}{\partial x_{j}} - \frac{\partial f}{\partial x_{j}} \frac{\partial g}{\partial \xi_{j}} \right)(x, \xi),
\]
where for simplicity we identify \((t, \tau)\) with \((x_{0}, \xi_{0})\) in this section. We state the following sufficient condition of (A.3), which is a generalized condition of that in [4].

**Proposition 4.1.** We have the following:
(i) When \( m_{k} = 1 \), the condition (A.3) is equivalent to the following:
\[
p_{m-1} + (i/2) r_{0}(\Lambda_{m-N+k}, \Lambda_{k}) |_{\tau = \lambda_{k}} \equiv 0 \mod \lambda_{k} - \lambda_{m-N+k}, \quad (4.1)
\]
where \( r_{0}(t, x, \tau, \xi) = p_{m}/(\tau - \lambda_{k})(\tau - \lambda_{m-N+k}) \).

(ii) When \( m_{k} = 2 \), if the following three conditions hold, then (A.3) is valid;
\[
p_{m-1}(t, x, \kappa_{k}(t, x, \xi), \xi) = 0, \quad (4.2)
\]
\[
\frac{\partial p_{m-1}}{\partial \tau} |_{\tau = \lambda_{k}} \equiv \{\Lambda_{m-N+k}, \Lambda_{k}\} |_{\tau = \lambda_{k}} \equiv 0 \mod \lambda_{k} - \lambda_{m-N+k}, \quad (4.3)
\]
\[
p_{m-2} - \sum_{j=0}^{n} p_{m-1,j}/2 + \sum_{l,j=0}^{n} p_{m,l}/8
\]
\[
- \sum_{j=0}^{n} r_{0}(\{\Lambda_{m-N+k}, \Lambda_{k}\} \Lambda_{k,j} - \{\Lambda_{m-N+k,j}, \Lambda_{k}\} \Lambda_{k}^{(j)})/4 \equiv 0 \mod \lambda_{k} - \lambda_{m-N+k}, \quad (4.4)
\]
where \( f_{\beta}^{(a)} = (iD_{x}^{a})D_{\xi}^{\beta}f(x, \xi) \) and \( r_{0}(t, x, \tau, \xi) = p_{m}/(\tau - \lambda_{k})^{2}(\tau - \lambda_{m-N+k}) \).

**Proof.** (i) Since \( R_{0}\Lambda_{k} \), where \( R_{0} \) is a pseudo-differential operator with the symbol \( p_{m}/(\tau - \lambda_{k}) \), satisfies the condition (A.3), we may show that the principal symbol \( r(x, \xi) \) of \( P - R_{0}\Lambda_{k} \) satisfies the condition: \( r |_{\tau = \lambda_{k}} \equiv 0 \mod \lambda_{k} - \lambda_{m-N+k} \). By an easy computation we see that
\[
(r - p_{m-1}) - \frac{i}{2} r_{0}(\Lambda_{m-N+k}, \Lambda_{k}) |_{\tau = \lambda_{k}} \equiv 0 \mod \lambda_{k} - \lambda_{m-N+k}.
\]
This implies (4.1).
(ii) In this case if we denote the principal symbol of $P - R_0 \Delta_k^2$ by $r(t, x, \tau, \xi)$, where $R_0$ is a pseudo-differential operator defined by the symbol $p_m/(\tau - \lambda_k)^2$, then $r$ is written by the following form:

$$r = p_{m-1}^s + (ir_0(\Lambda_{m-N+k}, \Lambda_k) + A\Lambda_{m-N+k})\Lambda_k,$$  \hspace{1cm} (4.5)

where $A$ is some positively homogeneous function of degree $m - 3$. Thus by (4.2) and (4.5) we can denote $P$ by $R_0 \Lambda_k^2 + R_1 \Lambda_k + R_2$, where $R_j$ is a pseudo-differential operator of order $m - 2$ and the principal symbol is written by $r_j(t, x, \tau, \xi)$. By (4.5) if we assume (4.3), then we have $r_{1|\tau=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}$. 

For simplicity we denote the homogeneous symbol of order $i$ of a pseudo-differential operator $Q(t, x, D_t, D_x)$ by $\sigma_i(Q)$. Then we have

$$r_{2|\tau=\lambda_k} = p_{m-2} - \sigma_{m-2}(R_0 \Lambda_k^2) - \sum_{j=0}^{n} r_{j|\tau=\lambda_k}(\Lambda_{k,j}^q)^i - 2^q \sigma_{m-2}(R_0 \Lambda_k^2) / 2,$$  \hspace{1cm} (4.6)

Since (4.3) holds and $r_1 \Lambda_k = p_{m-1} - \sigma_{m-1}(R_0 \Lambda_k^2)$, we have

$$\sum_{j=0}^{n} \left( r_{j|\tau=\lambda_k}(\Lambda_{k,j}^q) - \left( p_{m-1} - \sigma_{m-1}(R_0 \Lambda_k^2) \right) / 2 \right) \equiv 0 \mod \lambda_k - \lambda_{m-N+k}.$$  \hspace{1cm} (4.7)

Therefore we have

$$r_{2|\tau=\lambda_k} \equiv p_{m-2} - \sum_{j=0}^{n} p_{m-1,j}^q / 2 + \sum_{j=0}^{n} \sigma_{m-1}(R_0 \Lambda_k^2) / 2 - \sigma_{m-2}(R_0 \Lambda_k^2) |_{\tau=\lambda_k} \mod \lambda_k - \lambda_{m-N+k}.$$  \hspace{1cm} (4.7)

Here

$$\sigma_{m-1}(R_0 \Lambda_k^2) = \sum_{j=0}^{n} (r_0 \Lambda_k^q \Lambda_{k,j}^q + 2r_0 \Lambda_{k,j}^q \Lambda_k^q),$$  \hspace{1cm} (4.8)

$$\sigma_{m-2}(R_0 \Lambda_k^2) |_{\tau=\lambda_k} = \sum_{i,j=0}^{n} (r_0 \Lambda_{k,i}^q \Lambda_{k,j}^q + \Lambda_k^q \Lambda_{k,j}^q) + r_0 \Lambda_{k,i}^q \Lambda_{k,j}^q) |_{\tau=\lambda_k} \mod \lambda_k - \lambda_{m-N+k}.$$  \hspace{1cm} (4.9)

By taking care of (4.8), (4.9) and (4.3), add $\sum_{i,j=0}^{n} p_{m-1,j}^q / 2 |_{\tau=\lambda_k}$ to the right hand side of (4.7). Then we have that $r_{2|\tau=\lambda_k}$ is equal to the left hand side of (4.4) mod $\lambda_k - \lambda_{m-N+k}$. This completes the proof of Proposition 4.1.

REFERENCES


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