THE CAUCHY PROBLEM FOR HYPERBOLIC OPERATORS
WITH VARIABLE MULTIPLE CHARACTERISTICS

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Abstract. Let $P(t, x, D_t, D_x)$ be a hyperbolic differential operator with the
principal symbol $p_m(t, x, t, Q)$. We assume that $P_m$ is denoted by $\prod_{i=1}^{s} \frac{\partial}{\partial t} - (\lambda_i - \lambda_j)_{k=1, \ldots, s}$, where $\lambda_i(t, x, \xi) \in C^\infty([0, T] \times \mathbb{R}^n \times (\mathbb{R}^n \setminus 0))$. Under a generalized condition of E. E. Levi, we shall show that
the Cauchy problem $Pu = f$ in $[0, T] \times \mathbb{R}^n$, $\partial_t u|_{t=0} = g_1, \ldots, g_m$ is well posed. When $m_j = 1$ ($j = 1, \ldots, s$), our result coincides those of
Ohya and Petkov.

1. Statement of the result. We shall consider the following differential
operator:

$$P(t, x, D_t, D_x) = \sum_{|a| < m} a_a(t, x) D_t^{a_t} D_x^{a_x},$$

where $(t, x) \in [0, T] \times \mathbb{R}^n$ $(0 < T < \infty)$, $D_t a(t, x) = -a(t, x)$ and $a_a(t, x)$ belongs to $C_0^\infty([0, T] \times \mathbb{R}^n)$, which consists of all functions having that these
arbitrary derivations are bounded in $[0, T] \times \mathbb{R}^n$. Let $(t, \xi)$ be the covariable
of $(t, x)$. Then we define the following symbols:

$$p_k(t, x, \tau, \xi) = \sum_{|a| = k} a_a(t, x) \tau^{a_t} \xi^{a_x} \quad (k = 0, \ldots, m).$$

First we shall impose the following assumptions on $p_m$:

$$p_m(t, x, \tau, \xi) = \prod_{i=1}^{s} (\tau - \lambda_i)^{m_i} \prod_{j=s+1}^{m} (\tau - \lambda_j), \quad (A.1)$$

where $m_1 \geq m_2 \geq \cdots \geq m_s$, $N = \sum_{j=1}^{s} m_j$ and all functions $\lambda_j(t, x, \xi) (j = 1, \ldots, m - N + s)$ are real and positively homogeneous of degree 1 with
respect to $\xi$ and belong to $C_0^\infty([0, T] \times \mathbb{R}^n \times S^{n-1})$ if $\xi \in S^{n-1}$. Here $S^{n-1}$ is
the unit sphere of $\mathbb{R}^n$.\footnote{Received by the editors May 13, 1977 and, in revised form, August 8, 1977.}

$$p_m(t, x, \tau, \xi) \in [0, T] \times \mathbb{R}^n \times S^{n-1},$$

where $C$ is a positive constant.

Throughout this note, the symbols of pseudo-differential operators are
elements of $B([0, T] \times \mathbb{R}^n \times S^n)$ or $B([0, T] \times \mathbb{R}^n \times S^{n-1})$ if $(\tau, \xi) \in S^n$ or

\[ \xi \in S^{n-1}. \text{ Moreover we use the following notation. For symbols } a(t, x, \xi), \]
\[ b(t, x, \xi) \text{ and } A(t, x, \tau, \xi), A_{\tau=\xi} \equiv 0 \mod b \text{ means that there exists a symbol } \]
\[ c(t, x, \xi) \text{ such that} \]
\[ A(t, x, a(t, x, \xi), \xi) = c(t, x, \xi)b(t, x, \xi). \]

For the lower order terms of \( P \) we assume the following:

(A.3) For any \( k \) \((k = 1, \ldots, s)\) we can denote \( P(t, x, D_t, D_x) \) by the following form:

\[ P = \sum_{l=0}^{m_k} Q_{k,l}(t, x, D_t, D_x)(\Lambda_k(t, x, D_t, D_x))^l. \quad (1.1) \]

Here \( \Lambda_k \) is the pseudo-differential operator defined by the symbol \( \tau - \lambda_k(t, x, \xi) \) and \( Q_{k,l} \) \((l = 0, \ldots, m_k)\) is a pseudo-differential operator of order \( m - m_k \), whose principal symbol \( q_{k,l}(t, x, \tau, \xi) \) has the following property:

\[ q_{k,l,\tau=\xi} \equiv 0 \mod \lambda_k - \lambda_m - N + k. \quad (1.2) \]

Clearly if \( \lambda_k = \lambda_{m-N+k} \), then the above condition (A.3) is that of E. E. Levi.

In the final part of this note we denote (A.3) by the condition with respect to \( p_k \), when \( m_k = 1 \) or 2.

For a nonnegative integer \( k \) and \( s \in \mathbb{R} \) the function space \( C^k([0, T]; H_s(R^n)) \) consists of functions such that \( D^j u(t) \) \((j = 0, \ldots, k)\) exists as an element of \( H_{s-j}(R^n) \) and is continuous on the topology of \( H_{s-j}(R^n) \). We use the following norm:

\[ \| u(t) \|_{s,k}^2 = \sum_{j=0}^{k} \| D^j u(t) \|_{s-j}^2 \]

where \( \| \cdot \|_{s-j} \) is the usual norm of \( H_{s-j}(R^n) \).

Now we can state our theorem.

**THEOREM.** Let \( P(t, x, D_t, D_x) \) be a differential operator of order \( m \). If \( P \)

satisfies the assumptions (A.1), (A.2) and (A.3), then the Cauchy problem \( Pu = f \) in \([0, T] \times R^n, D_t^{m-1}u|_{t=0} = g_j \) \((j = 0, \ldots, m - 1)\) is well posed, i.e., for \( f \in C^{k-m+m_1+1}([0, T]; H_{s-m+m_1+1}(R^n)) \) and \( g_j \in H_{s-j+m_1}(R^n) \) there exists a unique solution \( u(t, x) \in C^k([0, T]; H_s(R^n)) \) such that

\[ \| u(t) \|_{s,k} \leq C \left( \sum_{j=0}^{m-1} \| g_j \|_{s-j+m_1} + \| f(0) \|_{s-m+m_1,k-m+m_1} \right. \]

\[ + \int_0^t \| f(r) \|_{s-m+m_1+k-m+m_1+1} dr \right), \]

where \( k > m - m_1 - 1, \| f(0) \|_{s-m+m_1-1} = 0 \) and \( t \in [0, T] \).

This Theorem is the same as those of [3] and [4] when \( m_j = 1 \) \((j = 1, \ldots, s)\). Under a different situation, in [1] they consider the Cauchy problem of a triple case.
2. Reform of the condition (A.3). In this section, we state an equivalent condition of (A.3).

Taking care of multiplicities of the roots \( \lambda_j \) \((j = 1, \ldots, s)\), we can denote \( p_m(t, x, \tau, \xi) \) by

\[
(\tau - \lambda_{m-N+s}) \cdots (\tau - \lambda_{r+1})\Phi_\mu \cdots \Phi_1^{r_s},
\]

where \( \Phi_\nu(t, x, \tau, \xi) \) \((\nu = 1, \ldots, \mu)\) is a polynomial of degree \( s \) with respect to \( \tau \) and equal to \( \prod_{i=1}^{s}(\tau - \lambda_i) \). Here \( m_{s+1} = \cdots = m_s \) \((\nu = 1, \ldots, \mu)\).

Remark that \( s_1 < s_2 < \cdots < s_\mu = s \), \( m_1 = \sum_{s_i - 1} N_p \) and denote \( N_p \) by \( s_i n_p \).

We introduce a product pseudo-differential operator \( \Phi(t, x, D_t, D_x) = (\Lambda_s \cdots \Lambda_1)(t, x, D_t, D_x) \). Then we denote \( \Delta_j(t, x, D_t, D_x) \) of order \( j \) \((j = 0, \ldots, \mu)\) by \( \Delta_0 = 1, \Delta_1 = \Lambda_1, \ldots, \Delta_\mu = \Phi^{s_\mu} \cdots \Phi_1^{s_1}, \ldots, \Delta_{m-N+k} = \Lambda_{s+k} \cdots \Lambda_{s+1} \cdots N_s \cdots \Lambda_{s+1} \cdots N_s \cdots \Lambda_{s+1} \cdots N_s \), where \( \Delta_j = \Lambda_s \cdots \Lambda_1 \Phi_\nu^{r_\mu} \cdots \Phi_1^{r_1} \) if \( j = (N_1 + \cdots + N_{\nu-1}) + s_\nu + \delta \) \((\nu = 0, \ldots, s, \delta = 0, \ldots, s_{\nu-1})\). Then we have the following:

**Proposition 2.1.** Let \( P(t, x, D_t, D_x) \) be a differential operator which satisfies the conditions (A.1) and (A.2). Then the condition (A.3) is equivalent to the following statement. We can denote \( P \) by

\[
P(t, x, D_t, D_x) = \sum_{i=0}^{m_1} Q_i(t, x, D_t, D_x) \Delta_{I(i)}, \tag{2.1}
\]

where if \( i = n_\mu + \cdots + n_{s+1} + \sigma \) \((1 < \sigma < n_s)\), then \( I(i) = N_1 + \cdots + N_\sigma \) and \( Q_i \) \((i = 0, \ldots, m_1)\) is a pseudo-differential operator of order \( M_i = m - i - I(i) \) and differential operator of \( t \). Moreover the principal symbol \( q_i(t, x, \tau, \xi) \) of \( Q_i \) satisfies the following condition:

\[
q_i |_{\tau = \lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k} \text{ if } k \leq s_\nu. \tag{2.2}
\]

Since any pseudo-differential operator of order \( l \) \((l < m)\) which is a differential operator of \( t \) is represented by \( \Delta_0, \ldots, \Delta_m \), we have the following:

**Proposition 2.2.** Let \( P \) be a differential operator which satisfies the conditions (A.1), (A.2), (2.1) and (2.2). Then \( P \) is denoted by

\[
P = \sum_{i=0}^{m_1} \sum_{j=0}^{M_i} R_{ij}(t, x, D_x) \Delta_{I(i)+j}, \tag{2.3}
\]

where \( R_{ij} \) is a pseudo-differential operator of order \( M_i - j \) whose principal symbol is \( r_{ij}(t, x, \xi) \), and \( r_{ij} \) have the following property:

\[
\sum_{j=0}^{k-1} r_{ij} \Delta_j \cdots \Lambda_1 |_{\tau = \lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \text{ if } k \leq s_\nu. \tag{2.4}
\]

if \( i = n_\mu + \cdots + n_{s+1} + \sigma \) \((1 < \sigma < n_s)\).

3. Reduction to a first order system and the proof of the Theorem. Since the proof of the Theorem is inferred on the analogy of a simple case, we assume that \( s = 2, m_1 = 2 \) and \( m_2 = 1 \). Thus by (2.3) our considered operator...
$P(t, x, D_t, D_x)$ in (1.1) is denoted by

$$
P(t, x, D_t, D_x) = \Delta_m + \sum_{i=1}^{m-i} \sum_{j=0}^{m-i} R_j^{m-i}\Delta_n
$$

where $R_j^{m-i}$ is of order $m - i - j$. The condition (2.4) says that

$$
R_{i-1}^{m-i}(t, x, D_x) = 0, \quad (3.1)
$$

$$
\rho_0^{m-1}(t, x, D_x) = \rho_0^{m-2}(t, x, D_x) = 0 \mod \lambda_1 - \lambda_{m-2}, \quad (3.2)
$$

$$
(r_1^{m-1} + r_2^{m-1}(\lambda_2 - \lambda_1))(t, x, \xi) \equiv 0 \mod \lambda_2 - \lambda_{m-1}. \quad (3.3)
$$

We denote a pseudo-differential operator with the symbol $|\xi|$ by the term $A$ and define a column vector

$$
U = \left( A_{m-3}u, A_{m-3}u, A_{m-4}u, A_{m-4}u, \ldots, A_{m-2}u, A_{m-1}u \right) \quad (3.4)
$$

and $F = \left( 0, \ldots, 0, f \right)$. Then by (3.1) the equation $Pu = f$ becomes the following first order system:

$$
MU = (D_t - A(t, x, D_x))U + B(t, x, D_x)U = F,
$$

where $B$ is of order 0 and $A$ is a first order pseudo-differential operator with the symbol

$$
a(t, x, \xi) = \begin{pmatrix}
\lambda_1, 0, \\
\lambda_1, |\xi|, 0, \\
\lambda_2, 0, \lambda_3, |\xi|, \\
\ldots, \\
\lambda_{m-2}, |\xi|, \\
a_m, a_{m-1}, a_{m-2}, a_{m-3}, 0, 0, 0, \ldots, 0, \lambda_{m-1}
\end{pmatrix}
$$

where $a_m = - \rho_0^{m-2}(t, x, \xi/|\xi||\xi|$, $a_{m-1} = - \rho_0^{m-1}(t, x, \xi/|\xi||\xi|$, $a_{m-2} = - \rho_2^{m-1}(t, x, \xi/|\xi||\xi|$. Then $a$ has the following property.

**Proposition 3.1.** There exists a nonsingular matrix $N(t, x, \xi)$ whose components are homogeneous of degree 0 and belong to $\mathcal{C}([-T, T], R^n \times (R^n \setminus 0))$ such that

$$(aN)(t, x, \xi) = (ND)(t, x, \xi),$$

where $D = (d_{ij})$ is a $m \times m$ diagonal matrix with $d_{11} = d_{22} = \lambda_1$ and $d_{ii} = \lambda_{i-1}$ $(i = 2, \ldots, m)$. 

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PROOF. Since clearly an eigenvalue of \( a(t, x, \xi/|\xi|) \) is \( \lambda_j = \lambda_j(t, x, \xi/|\xi|) \) 
\((j = 1, \ldots, m - 1)\), we shall seek an eigenvector of \( \tilde{\lambda}_j \). The equation
\[
(\mu I - a(t, x, \xi/|\xi|))n = 0,
\]
where \( n = (n_1, \ldots, n_m) \), is equivalent to the following:
\[
(\mu - \tilde{\lambda}_1)n_1 = 0, \quad (\mu - \tilde{\lambda}_1)n_2 - n_3 = 0, \quad (\mu - \tilde{\lambda}_2)n_3 = 0,
(\mu - \tilde{\lambda}_3)n_4 = n_5, \ldots, (\mu - \tilde{\lambda}_{m-2})n_{m-1} = n_m,
(\mu - \tilde{\lambda}_{m-1})n_m + \tilde{r}_0^{m-2}n_1 + \tilde{r}_1^{m-1}n_2 + \tilde{r}_2^{m-1}n_3 = 0,
\]
(3.5)
where \( \tilde{r}_j^{m-i}(t, x, \xi) = r_j^{m-i}(t, x, \xi/|\xi|) \). Since \( \tilde{r}_1^{m-1}n_2 + \tilde{r}_2^{m-1}n_3 = (\tilde{r}_1^{m-1} + \tilde{r}_2^{m-1}(\mu - \lambda_1))n_2 \), for example in fact we can take the following matrix \( N(t, x, \xi) \) such that each column of \( N \) is an eigenvector of an eigenvalue \( \tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_{m-1} \) respectively in turn;
\[
N(t, x, \xi) = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & (\lambda_2 - \lambda_1)^{-1} & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
n_{m-11} & n_{m-12} & n_{m-13} & \cdots & 0 & \vdots \\
n_{m1} & n_{m2} & n_{m3} & \cdots & 1
\end{pmatrix}
\]
where
\[
n_{m1} = \tilde{r}_0^{m-2}/(\tilde{\lambda}_1 - \tilde{\lambda}_m), \quad n_{m-11} = \tilde{r}_0^{m-2}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1})(\tilde{\lambda}_1 - \tilde{\lambda}_{m-2}),
n_{m2} = \tilde{r}_1^{m-1}/(\tilde{\lambda}_1 - \tilde{\lambda}_m), \quad n_{m-12} = \tilde{r}_1^{m-1}/(\tilde{\lambda}_1 - \tilde{\lambda}_{m-1})(\tilde{\lambda}_1 - \tilde{\lambda}_{m-2}),
n_{m3} = (\tilde{r}_1^{m-1} + \tilde{r}_2^{m-1}(\tilde{\lambda}_2 - \tilde{\lambda}_1))/(\tilde{\lambda}_2 - \tilde{\lambda}_{m-1}), \quad n_{m-13} = n_{m3}/(\tilde{\lambda}_2 - \tilde{\lambda}_{m-2}).
\]
These are well defined by (A.2), (3.2) and (3.3) and \( * \) are easily inductively determined from (A.2) and (3.5). Clearly the matrix \( N(t, x, \xi) \) is positively homogeneous of degree 0 and a nonsingular matrix in \([0, T] \times \mathbb{R}^n \). This completes the proof of Proposition 3.1.

By Proposition 3.1 the following fact is well known (see [2]), which guarantees the validity of our theorem.

PROPOSITION 3.2. For any nonnegative integer \( k \) and \( s \in \mathbb{R} \) if \( F(t, x) \in C^k([0, T]; H_s(\mathbb{R}^n)) \) and \( U_0(x) \in H_s(\mathbb{R}^n) \); then there exists a unique solution \( U(t, x) \in C^k([0, T]; H_s(\mathbb{R}^n)) \) of the Cauchy problem for \( M \) with initial data on \( t = 0 \) such that
\[
|||U(t)|||_{s,k} \leq C\left(|||U(0)|||_{s,k} + \int_0^1 |||MU(r)|||_{s,k} dr\right).
\]

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where $C$ does not depend on $U(t, x)$ and $t \in [0, T]$.

Remark. For a general case the vector $U$ of (3.4) becomes the following:

$$U = (\Lambda^{m-1} \Delta u, \ldots, \Lambda^{m-N-1} \Delta u, \ldots, \Lambda \Delta_{-2}, \Lambda \Delta_{-1})$$

where $a_j = m - (n_x + \cdots + n_y) + \sigma - j - 1$ if $j = N_1 + \cdots + N_{s-1} + s \sigma + \delta$ ($\sigma = 0, \ldots, n_x - 1, \delta = 0, \ldots, s_y - 1$).

4. Some sufficient condition of (A3) when $m_k = 1$ or 2. In this section we shall denote (A3) by the conditions with respect to $p_k$ when $m_k = 1$ or 2. We denote the subprincipal symbol of $P$ by

$$p_{m-1}^s(x, \xi) = p_{m-1} + (i/2) \sum_{j=0}^{n} \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}$$

and the Poisson bracket of $f(x, \xi)$ and $g(x, \xi)$ by

$$\{f, g\}(x, \xi) = \sum_{j=0}^{n} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right)(x, \xi),$$

where for simplicity we identify $(t, \tau)$ with $(x_0, \xi_0)$ in this section. We state the following sufficient condition of (A3), which is a generalized condition of that in [4].

**Proposition 4.1.** We have the following:

(i) When $m_k = 1$, the condition (A3) is equivalent to the following:

$$p_{m-1}^s + (i/2) r_0(\Lambda_{m-N+k}, \Lambda_k)|_{r=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}, \tag{4.1}$$

where $r_0(t, x, \tau, \xi) = p_m/(\tau - \lambda_k)(\tau - \lambda_{m-N+k})$.

(ii) When $m_k = 2$, if the following three conditions hold, then (A3) is valid:

$$\{f, g\}(x, \xi) = \sum_{j=0}^{n} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right)(x, \xi),$$

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where $r_0(t, x, \tau, \xi) = p_m/(\tau - \lambda_k)(\tau - \lambda_{m-N+k})$.

**Proof.** (i) Since $R_0 \Lambda_k$, where $R_0$ is a pseudo-differential operator with the symbol $p_m/(\tau - \lambda_k)$, satisfies the condition (A3), we may show that the principal symbol $r(t, \xi)$ of $P - R_0 \Lambda_k$ satisfies the condition: $r|_{r=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}$. By an easy computation we see that

$$(r - p_{m-1}^s) - \frac{i}{2} r_0(\Lambda_{m-N+k}, \Lambda_k)|_{r=\lambda_k} \equiv 0 \mod \lambda_k - \lambda_{m-N+k}.$$
(ii) In this case if we denote the principal symbol of $P - R_0 \Lambda_k^2$ by $r(t, x, \tau, \xi)$, where $R_0$ is a pseudo-differential operator defined by the symbol $p_m/(\tau - \lambda_k)^2$, then $r$ is written by the following form:

$$r = p_{m-1}^* + (ir_0(k \Lambda_{m-N+k}, k) + A \Lambda_{m-N+k}) \Lambda_k,$$  

(4.5)

where $A$ is some positively homogeneous function of degree $m - 3$. Thus by (4.2) and (4.5) we can denote $P$ by $R_0 \Lambda_k^2 + R_1 \Lambda_k + R_2$, where $R_j$ is a pseudo-differential operator of order $m - 2$ and the principal symbol is written by $r_j(t, x, \tau, \xi)$. By (4.5) if we assume (4.3), then we have $r_1(r \equiv 0 \mod \lambda_k - \lambda_{m-N+k}$.

For simplicity we denote the homogeneous symbol of order $i$ of a pseudo-differential operator $Q(t, x, D_t, D_x)$ by $\sigma_i(Q)$. Then we have

$$r(2r \equiv p_{m-2} - \sigma_{m-2}(R_0 \Lambda_k^2) - \sum_{j=0}^n r_j^{(j)} \Lambda_{k,j} \equiv \lambda_k, \quad (4.6)$$

Since (4.3) holds and $r_1 \Lambda_k = p_{m-1} - \sigma_{m-1}(R_0 \Lambda_k^2)$, we have

$$\sum_{j=0}^n \left( r_j^{(j)} \Lambda_{k,j} - (p_{m-1} - \sigma_{m-1}(R_0 \Lambda_k^2))^{(j)}/2 \right) \equiv 0 \mod \lambda_k - \lambda_{m-N+k}.$$  

Therefore we have

$$r(2r \equiv p_{m-2} - \sum_{j=0}^n p_j^{(j)}/2 + \sum_{j=0}^n \sigma_{m-1}(R_0 \Lambda_k^2)^{(j)}/2$$

$$- \sigma_{m-2}(R_0 \Lambda_k^2) \equiv \lambda_k - \lambda_{m-N+k}. \quad (4.7)$$

Here

$$\sigma_{m-1}(R_0 \Lambda_k^2) = \sum_{j=0}^n \left( r_0^{(j)}(\Lambda_{k,j} \Lambda_{k,j} + 2r_0^{(j)} \Lambda_{k,j} \Lambda_k) \right), \quad (4.8)$$

$$\sigma_{m-2}(R_0 \Lambda_k^2) \equiv \sum_{i,j=0}^n \left( r_0^{(i)}(\Lambda_{k,i} \Lambda_{k,j} + \Lambda_k^{(i)} \Lambda_{k,j}) + r_0^{(i)}(\Lambda_{k,i} \Lambda_{k,j}) \right) \equiv \lambda_k - \lambda_{m-N+k} \quad (4.9)$$

By taking care of (4.8), (4.9) and (4.3), add $\sum_{i,j=0}^n p_i^{(j)}/8 \equiv \lambda_k$ to the right hand side of (4.7). Then we have that $r(2r \equiv \lambda_k$ is equal to the left hand side of (4.4) mod $\lambda_k - \lambda_{m-N+k}$. This completes the proof of Proposition 4.1.

REFERENCES


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