ON ROBINSON'S $\frac{1}{2}$ CONJECTURE

ROGER W. BARNARD

Abstract. In 1947, R. Robinson conjectured that if $f$ is in $S$, i.e. a normalized univalent function on the unit disk, then the radius of univalence of $[zf(z)]'/2$ is at least $\frac{1}{2}$. He proved in that paper that it was at least $.38$. The conjecture has been shown to be true for most of the known subclasses of $S$. This author shows through use of the Grunski inequalities, that the minimum lower bound over the class $S$ lies between .49 and .5.

Introduction. Let $\mathcal{A}$ denote the class of analytic functions on the unit disk $U = \{z: |z| < 1\}$. Let $S$ denote the univalent functions $f$ in $\mathcal{A}$ normalized by $f(0) = 1 - f'(0) = 0$. Denote by $K$, $S^*$, $C$, and $Sp$ the standard subclasses of $S$ consisting of functions that are convex starlike, close to convex and spirallike respectively. For a subclass $X$ (possibly a singleton) of $\mathcal{A}$ let $r_S(X)$ denote the minimum radius of univalence over all functions $f$ in $X$. We use corresponding notation for the other subclasses of $S$. For example $r_{S^*}(X)$ denotes the minimum radius of starlikeness over all functions $f$ in $X$.

For a function $f$ in $S$ define the operator $\Gamma: S \to \mathcal{A}$ by $\Gamma f = (zf)'$. In 1947 R. Robinson [10] considered the problem of determining $r_S[\Gamma(S)]$. Robinson observed that for each $f$ in $S$, $[\Gamma(f)]' \neq 0$ for $|z| < \frac{1}{2}$. He also noted that for the Koebe function $k$, $k(z) = z(1 - z)^{-2}$, $r_S(k) = r_{S^*}(k) = \frac{1}{2}$, which implies $r_S[\Gamma(S)] < \frac{1}{2}$. He in fact conjectured that $r_S[\Gamma(S)] = \frac{1}{2}$. He was able to show that $r_{S^*}[\Gamma(S)] > .38$.

There have been a number of papers (e.g. [2], [3], [6], [7], [8]) on the connection between the operator $\Gamma$ and various subclasses of $S$. In these papers it has been shown that

$$r_K[\Gamma(K)] = r_{S^*}[\Gamma(S^*)] = r_C[\Gamma(C)] = r_{Sp}[\Gamma(Sp)] = \frac{1}{2}$$

and that $\Gamma$ preserves Rogosinski’s class of typically real functions (not necessarily univalent) up to $|z| < \frac{1}{2}$. It was observed in [2] that with the exception of the result $r_{Sp}[\Gamma(Sp)] = \frac{1}{2}$ these results follow directly from the S. Ruscheweyh-T. Sheil-Small theory [11]. They proved that, except for Sp, convolution by convex functions preserves the above subclasses of $S$. In order to obtain the related results in [2] one need only observe that for $f(z) = \sum a_n z^n$,

$$\Gamma[f(z)] = h \ast f(z) = \sum [(n + 1)/2]z^n \ast f(z) = \sum [(n + 1)/2]a_n z^n$$
and that \( h(z) = (z - z^2/2)(1 - z)^{-2} \) is convex for \( |z| < \frac{1}{2} \). As was shown in [2] most of the results that had been obtained on generalizations of the operator \( \Gamma \) on subclasses of \( S \) can also be obtained in a similar manner by the appropriate modifications of \( h \). However, for the entire class \( S \), if we let \( r_0 = r_C(S) \approx .80 \) from [5], it appears that the easily obtained lower bound for \( r_S[\Gamma(S)] \) of \( r_0/2 \approx .41 \) is the most that can be obtained from the convolution operator method. It does show that \( .41 < r_S[\Gamma(S)] \leq \frac{1}{2} \). In the present note the author, through use of the Grunsky inequalities, is able to prove that \( .49 < r_S[\Gamma(S)] \leq \frac{1}{2} \).

**Proof of Main Result.** To find a lower bound for \( r_S[\Gamma(S)] \) we consider the nonvanishing of

\[
\frac{f(z) + zf'(z) - f(\xi) - \xi f'(\xi)}{z - \xi}.
\]

By use of the minimum principle we may assume \( |z| = |\xi| < r \). Since \( f \) is in \( S \) we may divide through by \( |f(z) - f(\xi)|/|z - \xi| \). Thus it suffices to find the largest \( r \) such that

\[
1 + \frac{zf'(z) - \xi f'(\xi)}{f(z) - f(\xi)} 
eq 0, \quad |z| = |\xi| < r. \tag{1}
\]

Consider for \( f \) in \( S \) the Grunsky coefficients defined by letting

\[
\log \frac{f(z) - f(\xi)}{z - \xi} = \sum_{n,m=0}^{\infty} d_{nm} z^n \xi^m. \tag{2}
\]

Putting \( \xi, z = 0 \) respectively, in (2) we obtain

\[
\log \frac{f(z)}{z} = \sum_{n=0}^{\infty} d_{0n} z^n, \quad \log \frac{f(\xi)}{\xi} = \sum_{m=0}^{\infty} d_{0m} \xi^m.
\]

Hence

\[
\log \frac{f(z) - f(\xi)}{z - \xi} = \log \frac{f(z)}{z} + \log \frac{f(\xi)}{\xi} + \sum_{n,m=1}^{\infty} d_{nm} z^n \xi^m. \tag{3}
\]

Although Grunsky's inequalities are usually stated in terms of the function \( F \), on \( |\xi| > 1 \) defined by \( F(\xi) = 1/f(1/\xi) \), it is more convenient for our purposes to express them directly in terms of \( f \) in \( S \). To do this, we observe, by letting \( z' = 1/z \), \( \xi' = 1/\xi \), that

\[
\log \frac{f(z) - f(\xi)}{z - \xi} = \log \frac{f(1/z') - f(1/\xi')}{f(1/z')f(1/\xi') z' \xi'[(1/z') - (1/\xi')]}
\]

\[
= \log \frac{1/f(1/z') - 1/f(1/\xi')}{z' - \xi'} = \log \frac{F(z') - F(\xi')}{z' - \xi'}
\]

\[
= \sum_{n,m=1}^{\infty} d_{nm} (z')^{-n} (\xi')^{-m} = \sum_{n,m=1}^{\infty} d_{nm} z^n \xi^m.
\]
Thus we can use the following form of Grunsky’s inequalities (see Pommerenke [9, p. 60]). For \( f \) in \( S \) and \( d_{nm} \) defined by (2) we have for arbitrary complex \( x_n \),

\[
\sum_{n=1}^{\infty} n \sum_{m=1}^{\infty} d_{nm} x_m^2 \leq \sum_{n=1}^{\infty} \frac{|x_n|^2}{n}
\]

(4)

provided the last series converges. Now, differentiating (3) with respect to \( z \) and \( \zeta \) we see from the uniform convergence of the series in (3) for \(|z| = |\zeta| < r < 1\) that

\[
\frac{zf'(z)}{f(z) - f(\zeta)} - \frac{z}{z - \zeta} = \frac{zf'(\zeta)}{f(z)} - 1 + \sum_{m,n=1}^{\infty} nd_{nm} z^n \zeta^m,
\]

and

\[
\frac{-\zeta f'(\zeta)}{f(z) - f(\zeta)} + \frac{\zeta}{z - \zeta} = \frac{\zeta f'(\zeta)}{f(\zeta)} - 1 + \sum_{n,m=1}^{\infty} md_{nm} z^n \zeta^m.
\]

Adding these two expressions and rearranging we obtain

\[
1 + \sum_{n,m=1}^{\infty} nd_{nm} z^n \zeta^m = \sum_{n,m=1}^{\infty} (n + m) d_{nm} z^n \zeta^m.
\]

(5)

Thus from (1) we need to find the largest \( r \) for which the right-hand side of (5), which we denote by \( T(z, \zeta) \), does not vanish for \(|z| = |\zeta| < r\). We have by the use of Schwarz’s inequality and (4) that

\[
\Re \{ T(z, \zeta) \} > \Re \left\{ \frac{zf'(z)}{f(z)} + \frac{\zeta f'(\zeta)}{f(\zeta)} \right\} - \left| \sum_{n,m=1}^{\infty} (n + m) d_{nm} z^n \zeta^m \right|
\]

\[
> 2 \min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - \left| \sum_{n=1}^{\infty} \sqrt{n} z^n \left( \sum_{m=1}^{\infty} d_{nm} \zeta^m \right) \right|
\]

\[
> 2 \min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - \left( \sum_{n=1}^{\infty} m^{2n} \right)^{1/2} \left( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} \zeta^m \right)^{1/2}
\]

\[
> 2 \min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - \left( \sum_{m=1}^{\infty} m^{2n} \right)^{1/2} \left[ \sum_{m=1}^{\infty} m^{2n} \sum_{n=1}^{\infty} d_{nm} \zeta^m \right]^{1/2}
\]

\[
> 2 \min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} - \left[ \frac{r^2}{(1 - r^2)^2} \right]^{1/2} \left( \sum_{n=1}^{\infty} \frac{r^{2n}}{n} \right)^{1/2}
\]
Since $f$ is in $S$ we have the well-known inequality
\[
\log \frac{zf'(z)}{f(z)} \leq \log \frac{1 + r}{1 - r}, \quad |z| = r < 1,
\]
where, in fact, for each real $\alpha$, and $z$, $|z| = r < 1$, there exists an $\alpha$ in $S$ such that
\[
\log \left[ \frac{zf'(z)}{f(z)} \right] = e^{i\alpha} \log \left[ \frac{(1 + r)/(1 - r)}{1 - r^2} \right].
\]
(see Jenkins [4, p. 110]). Thus, if we let $\log \left[ \frac{zf'(z)}{f(z)} \right] = e^{i\alpha} \log \left[ \frac{(1 + r)/(1 - r)}{1 - r^2} \right]$, we can assume $R = \log(1 + r)/(1 - r)$. In order to find the minimum of (6) for all $f$ in $S$ we consider
\[
\min_{f \in S} \min_{|z|=r} \Re \left\{ \frac{zf'(z)}{f(z)} \right\} = \min_{\Phi} \Re \{ \exp(e^{i\Phi}) \}
\]
\[
= \min_{\Phi} \left[ \exp(R \cos \Phi) \right] \left[ \cos(R \sin \Phi) \right].
\]
Thus, from (6), we need to find the largest $r$ for which
\[
\min_{\Phi} \left[ \exp(R \cos \Phi) \right] \left[ \cos(R \sin \Phi) \right] > \frac{r}{1 - r^2} \left[ -\log(1 - r^2) \right]^{1/2}. \quad (7)
\]
It is easy to see that the left-hand side of (7), call it $LS$, is a decreasing function of $r$ while the right-hand side of (7), call it $RS$, is an increasing function of $r$. A computer checked calculation shows that for $r = .490$, $RS < .3379$ while $LS > .3393$ where the minimum value occurs when $\Phi$ is approximately 2.5 radians. We note that for $r = .491$, $RS > .3398$. Thus, inequality (1) holds for all $f$ in $S$ and $r < .49$. It follows that $r_5[\Gamma(S)] > .49$.

**Bibliography**


**Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506**

**Current address:** Department of Mathematics, Texas Tech University, Lubbock, Texas 79409